Estimation and Inference in Boundary Discontinuity Designs: Distance-Based Methods Supplemental Appendix

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Abstract

This supplemental appendix presents more general theoretical results encompassing those reported in the paper, their theoretical proofs, and other technical results. It also provides additional numerical results.

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Contents

SA-1 Introduction **1**

SA-1.1 Notation and Definitions . . . 1

SA-2 Setup **2**

SA-2.1 Assumptions . . . 3

SA-2.2 Parameters . . . 4

SA-2.3 Estimators and Other Related Quantities . . . 5

SA-2.4 Higher-Order Signed-Distance Derivatives . . . 6

SA-2.5 Preliminary Lemmas . . . 7

SA-3 Boundary Average Treatment Effect Curve **9**

SA-3.1 Identification . . . 9

SA-3.2 Approximation Bias . . . 9

SA-3.3 Point Estimation and Approximate MSE Expansions . . . 10

SA-3.4 Distributional Approximation and Inference . . . 11

SA-4 Weighted Boundary Average Treatment Effect **14**

SA-5 Largest Boundary Average Treatment Effect **16**

SA-6 Imperfect Compliance **17**

SA-6.1 Second-Order Linearization . . . 18

SA-6.2 Fuzzy BATEC . . . 20

SA-6.3 Fuzzy WBATE . . . 22

SA-6.4 Fuzzy LBATE . . . 24

SA-7 Gaussian Strong Approximation **25**

SA-7.1 Definitions for Function Spaces . . . 25

SA-7.2 Multiplicative-Separable Empirical Process . . . 26

SA-7.3 Proof of Theorem SA-25 . . . 27

SA-8 Proofs **29**

SA-8.1 Proof of Lemma SA-1 . . . 29

SA-8.2 Proof of Lemma SA-2 . . . 30

SA-8.3 Proof of Lemma SA-3 . . . 32

SA-8.4 Proof of Lemma SA-4 . . . 33

SA-8.5 Proof of Lemma SA-5 . . . 35

SA-8.6 Proof of Theorem SA-1 . . . 37

SA-8.7 Proof of Theorem SA-2 . . . 37

SA-8.8 Proof of Theorem SA-3 . . . 39

SA-8.9 Proof of Theorem SA-4 . . . 41

SA-8.10 Proof of Theorem SA-5 . . . 41

SA-8.11 Proof of Theorem SA-6 . . . 42

SA-8.12 Proof of Theorem SA-7 . . . 43

SA-8.13 Proof of Theorem SA-8 . . . 45
SA-8.14 Proof of Theorem SA-9 . . . 49
SA-8.15 Proof of Theorem SA-10 . . . 50
SA-8.16 Proof of Lemma SA-6 . . . 52
SA-8.17 Proof of Lemma SA-7 . . . 52
SA-8.18 Proof of Theorem SA-11 . . . 54
SA-8.19 Proof of Theorem SA-12 . . . 54
SA-8.20 Proof of Theorem SA-13 . . . 55
SA-8.21 Proof of Theorem SA-14 . . . 55
SA-8.22 Proof of Theorem SA-15 . . . 55
SA-8.23 Proof of Theorem SA-16 . . . 55
SA-8.24 Proof of Lemma SA-8 . . . 56
SA-8.25 Proof of Theorem SA-17 . . . 56
SA-8.26 Proof of Theorem SA-18 . . . 57
SA-8.27 Proof of Theorem SA-19 . . . 57
SA-8.28 Proof of Theorem SA-20 . . . 57
SA-8.29 Proof of Lemma SA-9 . . . 58
SA-8.30 Proof of Lemma SA-10 . . . 58
SA-8.31 Proof of Theorem SA-21 . . . 59
SA-8.32 Proof of Theorem SA-22 . . . 59
SA-8.33 Proof of Theorem SA-23 . . . 60
SA-8.34 Proof of Theorem SA-24 . . . 60
SA-8.35 Proof of Theorem 6 . . . 60

SA-9 Simulation Study **63**
SA-9.1 Data-Generating Process . . . 63
SA-9.2 Calibration from the SPP Data . . . 65
SA-9.3 Monte Carlo Implementation . . . 67
SA-9.4 Results . . . 68

SA-1 Introduction

This supplemental appendix considers four generalizations relative to the setup in the paper.

1. The score variable $\mathbf{X}_i$ is d-dimensional, with the genuinely multivariate boundary-discontinuity case corresponding to $d \geq 2$.
2. *Weighted Boundary Average Treatment Effect* (WBATE).
3. *Largest Boundary Average Treatment Effect* (LBATE).
4. Imperfect compliance (fuzzy) designs.

In addition, this supplement presents technical results and proofs not covered in the main text due to space limitations.

The assumptions in the paper are specialized primitive conditions; after imposing these restrictions on the supplemental results, the mapping between this supplemental appendix and the paper is as follows.

- Theorem SA-1 $\implies$ Theorem 1.
- Theorem SA-2, in its sharp level case with $d = 2$, $\implies$ Theorem 2.
- Theorem SA-3, with $d = 2$, $\implies$ Theorem 3.
- Theorem SA-4 with $d = 2$ $\implies$ Theorem 4.
- Theorem SA-5 with $d = 2$ $\implies$ Theorem 5(i).
- Theorem SA-8 with $d = 2$ $\implies$ Theorem 5(ii).
- Section SA-8.35 $\implies$ Theorem 6.

The generalizations considered in this supplemental appendix are conceptually straightforward, but technically more involved. Section SA-3 presents the main distance-based identification, estimation, and inference results. Sections SA-4 and SA-5 present the corresponding WBATE and LBATE results for sharp designs. Section SA-6 extends all the results to fuzzy designs. Section SA-7 presents a new Gaussian strong approximation result that may be of independent interest.

SA-1.1 Notation and Definitions

For textbook references on empirical process, see van der Vaart and Wellner [1996], Dudley [2014], and Giné and Nickl [2016]. For textbook references on geometric measure theory, see Simon et al. [1984], Federer [2014], and Folland [2002]. Throughout the appendix, $\mathbf{x}$ denotes a generic point in $\mathscr{X}$ and, when $\mathbf{x} \in \mathscr{B}$, a generic boundary point; the main paper uses $\mathbf{b}$ and $\mathbf{b}_j$ for selected or discretized evaluation points on the boundary.

(i) *Polynomial Notation.* For scalar $u \in \mathbb{R}$, $\mathbf{r}_p(u) = (1, u, \ldots, u^p)^\top$, and $\mathbf{e}_0$ denotes the intercept selector satisfying $\mathbf{e}_0^\top \mathbf{r}_p(u) = 1$. More generally, $\mathbf{e}_j$ denotes the selector satisfying $\mathbf{e}_j^\top \mathbf{r}_p(u) = u^j$ for $j = 0, \ldots, p$.

(ii) *Norms.* For a vector $\mathbf{a} \in \mathbb{R}^k$, $\|\mathbf{a}\| = (\sum_{i=1}^k \mathbf{a}_i^2)^{1/2}$ and $\|\mathbf{a}\|_\infty = \max_{1\le i\le k} |\mathbf{a}_i|$. For a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\|\mathbf{A}\|_p = \sup_{\|\mathbf{x}\|_p=1} \|\mathbf{A}\mathbf{x}\|_p$, $p \in \mathbb{N} \cup \{\infty\}$, and $\lambda_{\min}(\mathbf{A})$ denotes its minimum eigenvalue when defined. For a function f on a metric space (S, d), $\|f\|_\infty = \sup_{s\in S} |f(s)|$, and $\operatorname{supp}(f)$ denotes the closure of $\{s \in S : f(s) \neq 0\}$. For a probability measure Q on $(\mathcal{S}, \mathscr{S})$ and $p \ge 1$, define $\|f\|_{Q,p} = (\int_{\mathcal{S}} |f|^p \,\mathrm{d}Q)^{1/p}$, and $Q(f) = \int f \,\mathrm{d}Q$. For a set $E \subseteq \mathbb{R}^d$, $\mathfrak{m}(E)$ denotes the Lebesgue measure of E.

(iii) *Empirical Process.* We use standard empirical-process notation: $\mathbb{E}_n[g(\mathbf{v}_i)] = n^{-1}\sum_{i=1}^n g(\mathbf{v}_i)$ and $\mathbb{G}_n[g(\mathbf{v}_i)] = n^{-1/2}\sum_{i=1}^n (g(\mathbf{v}_i) - \mathbb{E}[g(\mathbf{v}_i)])$. Let $(\mathcal{S}, d)$ be a semi-metric space. The covering number $N(\mathcal{S}, d, \varepsilon)$ is the minimal number of open d-balls of radius ε needed to cover $\mathcal{S}$. A $\mathbb{P}$-*Brownian bridge* is a mean-zero Gaussian random function $W_{\mathbb{P}}(f)$, $f \in L_2(\mathcal{X}, \mathbb{P})$, with covariance $\mathbb{E}[W_{\mathbb{P}}(f)W_{\mathbb{P}}(g)] = \mathbb{P}(fg) - \mathbb{P}(f)\mathbb{P}(g)$. A class $\mathscr{F} \subseteq L_2(\mathcal{X}, \mathbb{P})$ is $\mathbb{P}$-*pregaussian* if there is a version of $W_{\mathbb{P}}$ such that $W_{\mathbb{P}} \in C(\mathscr{F}; \rho_{\mathbb{P}})$ almost surely, where $\rho_{\mathbb{P}}(f, g) = (\|f - g\|_{\mathbb{P},2}^2 - (\mathbb{P}f - \mathbb{P}g)^2)^{1/2}$.

(iv) *Geometric Measure Theory.* For a set $E \subseteq \mathcal{X}$, the *De Giorgi perimeter of E relative to $\mathcal{X}$* is $\mathscr{L}(E) = \mathtt{TV}_{\{\mathbb{1}_E\},\mathcal{X}}$. For $d \in \mathbb{N}$ and $0 \le m \le d$, $\mathfrak{H}^m$ denotes m-dimensional Hausdorff measure, and integration against $\mathfrak{H}^m$ is understood in the standard Caratheodory construction. The Hausdorff dimension of A is denoted $\dim_{\mathfrak{H}}(A)$. A set $A \subseteq \mathbb{R}^d$ is k-rectifiable if it has Hausdorff dimension k and is covered up to a $\mathfrak{H}^k$-null set by countably many continuously differentiable maps from $\mathbb{R}^k$ to $\mathbb{R}^d$. A set $C \subseteq \mathbb{R}^d$ is a *rectifiable curve* if there exists a Lipschitz continuous function $\gamma : [0, 1] \to \mathbb{R}^d$ such that $C = \gamma([0, 1])$.

(v) *Bounds and Asymptotics.* For real sequences, $a_n = o(b_n)$ means $a_n/b_n \to 0$, and $a_n \lesssim b_n$ means $|a_n| \le C|b_n|$ eventually for a finite constant C. For random sequences, $a_n = o_{\mathbb{P}}(b_n)$ means $a_n/b_n \to_{\mathbb{P}} 0$, and $|a_n| \lesssim_{\mathbb{P}} |b_n|$ means $\limsup_{M\to\infty} \limsup_{n\to\infty} \mathbb{P}[|a_n/b_n| \ge M] = 0$.

(vi) *Distributions and Statistical Distances.* For $\boldsymbol{\mu} \in \mathbb{R}^k$ and a $k\times k$ positive definite matrix $\boldsymbol{\Sigma}$, $\mathsf{Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. $\mathsf{Uniform}([a, b])$ denotes the uniform distribution on $[a, b]$, $\mathsf{Bernoulli}(p)$ denotes the Bernoulli distribution with success probability p, and $\Phi(\cdot)$ denotes the standard Gaussian cumulative distribution function. For two distributions P and Q, $d_{\mathrm{KL}}(P, Q)$ denotes the KL-distance and $d_{\chi^2}(P, Q)$ denotes the χ^2 distance.

Throughout the appendix, d and the local-polynomial order p are fixed, with $p \ge 0$. All limits are taken with $h \to 0$ as $n \to \infty$. The genuinely multivariate boundary-discontinuity results are stated for $d \ge 2$; the BATEC results (pointwise and uniform over finite boundary sets), the LBATE results, and their fuzzy analogues recover the usual one-dimensional RD case when $d = 1$ under the analogous assumptions, interpreting $\mathscr{B}$ as a finite 0-rectifiable boundary and integrating with respect to $\mathfrak{H}^0$. Most of our results hold with fixed but small enough bandwidths; we do not make this distinction explicit to avoid overly complex statements and technical discussions.

SA-2 Setup

We partition $\mathcal{X}$ into two assignment areas, $\mathscr{A}_t \subset \mathbb{R}^d$ with $t \in \{0, 1\}$, representing the control and treatment regions, respectively. That is, $\mathcal{X} = \mathscr{A}_0 \cup \mathscr{A}_1$, where $\mathscr{A}_0$ and $\mathscr{A}_1$ are disjoint regions in $\mathbb{R}^d$. The observed outcome is $Y_i = \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_0)Y_i(0) + \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_1)Y_i(1)$. The boundary is $\mathscr{B} = \mathrm{bd}(\mathscr{A}_0) \cap \mathrm{bd}(\mathscr{A}_1) \subseteq \mathbb{R}^d$, where $\mathrm{bd}(\cdot)$ denotes the topological boundary of a set, and has effective dimension $d-1$. Without loss of generality, we assume that $\mathscr{B}$ belongs to the treatment region, that is, $\mathscr{B} \subseteq \mathscr{A}_1$ and $\mathscr{B} \cap \mathscr{A}_0 = \emptyset$.

The univariate distance score induced by the multidimensional score is

$$D_i(\mathbf{x}) = [\mathbb{1}(\mathbf{X}_i \in \mathscr{A}_1) - \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_0)]\mathscr{d}(\mathbf{X}_i, \mathbf{x}), \qquad \mathbf{x} \in \mathscr{B},$$

where $\mathscr{d}(\cdot,\cdot)$ denotes a distance function. For a deterministic point $\mathbf{z} \in \mathscr{X}$, write $D_{\mathbf{z}}(\mathbf{x}) = [\mathbb{1}(\mathbf{z} \in \mathscr{A}_1) - \mathbb{1}(\mathbf{z} \in \mathscr{A}_0)]\mathscr{d}(\mathbf{z}, \mathbf{x})$. Let $\mathscr{I}_0 = (-\infty, 0)$, $\mathscr{I}_1 = [0, \infty)$, and $\varsigma_t = 2t - 1$ for $t \in \{0, 1\}$. Thus, for $\mathbf{z} \in \mathscr{A}_t$, $D_{\mathbf{z}}(\mathbf{x}) = \varsigma_t \mathscr{d}(\mathbf{z}, \mathbf{x})$. Let $\mathbf{r}_p(u) = (1, u, \cdots, u^p)^\top$, and $K_h(u) = K(u/h)/h^d$, where $K(\cdot)$ is a univariate kernel and h is a bandwidth parameter. No symmetry of K is imposed, so the side-specific sign remains in both the polynomial argument and the kernel weight.

For later reference and for Assumption SA-2, let

$$\mathbf{\Psi}_{t,\mathbf{x}} = \mathbb{E}\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)^\top K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t)\Big], \qquad t \in \{0, 1\}.$$

SA-2.1 Assumptions

The data generating process assumption from the main paper generalizes as follows.

Assumption SA-1 (Data Generating Process)**.** *Let $t \in \{0, 1\}$.*

(i) $(Y_1(t), \mathbf{X}_1^\top)^\top, \ldots, (Y_n(t), \mathbf{X}_n^\top)^\top$ *are independent and identically distributed random vectors.*

(ii) *The distribution of $\mathbf{X}_i$ has compact support $\mathscr{X} \subseteq \mathbb{R}^d$ with nonempty interior and a Lebesgue density $f_X(\mathbf{x})$ that is continuous and bounded away from zero on $\mathscr{X}$.*

(iii) $\mu_t(\mathbf{x}) = \mathbb{E}[Y_i(t)|\mathbf{X}_i = \mathbf{x}]$ *has a $(p+1)$-times continuously differentiable extension to an open neighborhood of $\mathscr{X}$.*

(iv) $\sigma_t^2(\mathbf{x}) = \mathbb{V}[Y_i(t)|\mathbf{X}_i = \mathbf{x}]$ *is bounded away from zero and continuous on $\mathscr{X}$.*

(v) $\sup_{\mathbf{x}\in\mathscr{X}} \mathbb{E}[|Y_i(t)|^{2+v}|\mathbf{X}_i = \mathbf{x}] < \infty$ *for some $v \geq 2$.*

The boundary, distance, and kernel assumption from the main paper generalizes as follows.

Assumption SA-2 (Boundary, Distance, and Kernel)**.** *Let $t \in \{0, 1\}$.*

(i) $\mathscr{B} \subset \operatorname{int}(\mathscr{X})$*, and $\mathscr{B}$ is compact $(d-1)$-rectifiable, with $\mathfrak{H}^{d-1}(\mathscr{B})$ positive and finite.*

(ii) $\mathscr{d} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ *is a metric on $\mathbb{R}^d$ equivalent to the Euclidean distance, that is, there exist positive constants C_u and C_l such that $C_l \|\mathbf{x} - \mathbf{x}'\| \leq \mathscr{d}(\mathbf{x}, \mathbf{x}') \leq C_u \|\mathbf{x} - \mathbf{x}'\|$ for all $\mathbf{x}, \mathbf{x}' \in \mathscr{X}$.*

(iii) $K : \mathbb{R} \to [0, \infty)$ *is either compactly supported and Lipschitz continuous, or $K(u) = \mathbb{1}(u \in [-1, 1])$ and the distance balls $\{\mathbf{v} \in \mathscr{X} : \mathscr{d}(\mathbf{v}, \mathbf{x}) \leq r\}$, indexed by $(\mathbf{x}, r) \in \mathscr{X} \times \mathbb{R}_+$, form a VC class.*

(iv) $\liminf_{h\downarrow 0} \inf_{\mathbf{x}\in\mathscr{B}} \lambda_{\min}(\mathbf{\Psi}_{t,\mathbf{x}}) \gtrsim 1$.

(v) *For each $\mathbf{x} \in \mathscr{B}$ and $t \in \{0, 1\}$, the denominator in the Hausdorff-measure representation of $\theta_{t,\mathbf{x}}(r)$ below is positive and finite for all sufficiently small $|r| > 0$ with $r \in \mathscr{I}_t$.*

For $d = 2$, Assumption SA-2(i) holds whenever $\mathscr{B}$ is a compact rectifiable curve contained in the interior of $\mathscr{X}$, with positive and finite length. Assumption SA-2(ii) is specific to distance-based methods and requires the distance function to be equivalent, up to constants, to the Euclidean distance. In Assumption SA-2(iii), the extra VC condition appears only for the uniform kernel because its indexing sets are distance balls rather than rectangles: Lemma SA-1 gives primitive cases where it is automatic, including the Euclidean distance.

Assumption SA-2(iv) rules out boundary–distance–kernel configurations in which, after local rescaling by h, one side of the boundary receives too little kernel-weighted mass to guarantee that the distance-based local polynomial regression is well-defined uniformly over $\mathscr{B}$. Thus, the condition ensures that the local distance

design is sufficiently rich on each side of the boundary. Lemma SA-2 gives primitive sufficient conditions for the Euclidean-distance case, and Lemma SA-3 then shows that the sample Gram matrix is full rank in large samples. Assumption SA-2(v) is specific to distance-based identification: it keeps the Hausdorff-measure ratio defining $\theta_{t,\mathbf{x}}(r)$ well-defined near the boundary and is used for identification (Theorem SA-1); under additional smoothness, the same ratio is differentiated in Theorem SA-3. These conditions are stated uniformly over $\mathscr{B}$ in anticipation of our uniform estimation and inference results, but several can be relaxed to hold only pointwise in $\mathbf{x} \in \mathscr{B}$ for pointwise results.

Assumption 3 imposes regularity conditions on a weight function along the boundary.

Assumption SA-3 (Weight Function and Boundary)**.** *Let* $w : \mathscr{B} \to (0, \infty)$ *satisfy* $\sup_{\mathbf{x}\in\mathscr{B}} w(\mathbf{x}) < \infty$, $\inf_{\mathbf{x}\in\mathscr{B}} w(\mathbf{x}) > 0$, *and* $\int_{\mathscr{B}} w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}) < \infty$.

SA-2.2 Parameters

The *Boundary Average Treatment Effect Curve* (BATEC) is

$$\tau(\mathbf{x}) = \mathbb{E}[Y_i(1) - Y_i(0) | \mathbf{X}_i = \mathbf{x}], \qquad \mathbf{x} \in \mathscr{B}.$$

For each $t \in \{0, 1\}$, the induced conditional expectation based on univariate distance is

$$\theta_{t,\mathbf{x}}(r) = \mathbb{E}[Y_i | D_i(\mathbf{x}) = r] = \mathbb{E}[Y_i | \mathscr{d}(\mathbf{X}_i, \mathbf{x}) = |r|, \mathbf{X}_i \in \mathscr{A}_t], \qquad r \in \mathscr{I}_t, \quad \mathbf{x} \in \mathscr{B}.$$

More rigorously, for each $t \in \{0, 1\}$, and letting $S_{t,\mathbf{x}}(r) = \{\mathbf{v} \in \mathscr{X} : \mathscr{d}(\mathbf{v}, \mathbf{x}) = r, \mathbf{v} \in \mathscr{A}_t\}$ for $r \geq 0$ and $\mathbf{x} \in \mathscr{B}$,

$$\theta_{t,\mathbf{x}}(r) = \frac{\int_{S_{t,\mathbf{x}}(|r|)} \mu_t(\mathbf{v}) f_X(\mathbf{v})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{v})}{\int_{S_{t,\mathbf{x}}(|r|)} f_X(\mathbf{v})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{v})},$$

for $|r| > 0, \mathbf{x} \in \mathscr{B}, t \in \{0, 1\}$, and therefore (under our assumptions)

$$\theta_{t,\mathbf{x}}(0) = \lim_{r\to 0,\, r\in\mathscr{I}_t} \mathbb{E}[Y_i | \mathscr{d}(\mathbf{X}_i, \mathbf{x}) = |r|, \mathbf{X}_i \in \mathscr{A}_t] = \lim_{r\to 0,\, r\in\mathscr{I}_t} \frac{\int_{S_{t,\mathbf{x}}(|r|)} \mu_t(\mathbf{v}) f_X(\mathbf{v})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{v})}{\int_{S_{t,\mathbf{x}}(|r|)} f_X(\mathbf{v})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{v})}.$$

The induced signed-distance treatment-effect parameter is

$$\vartheta(\mathbf{x}) = \theta_{1,\mathbf{x}}(0) - \theta_{0,\mathbf{x}}(0), \qquad \mathbf{x} \in \mathscr{B}.$$

Theorem SA-1 shows that $\vartheta(\mathbf{x}) = \tau(\mathbf{x})$ under Assumptions SA-1(i)-(iii) and SA-2(i), (ii), and (v).

We also consider the parameters

$$\vartheta_{\texttt{WBATE}} = \frac{\int_{\mathscr{B}} \vartheta(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})}{\int_{\mathscr{B}} w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})} \qquad \text{and} \qquad \vartheta_{\texttt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \vartheta(\mathbf{x}).$$

Theorem SA-1 implies that

$$\vartheta_{\texttt{WBATE}} = \tau_{\texttt{WBATE}} = \frac{\int_{\mathscr{B}} \tau(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})}{\int_{\mathscr{B}} w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})}$$

is the *Weighted Boundary Average Treatment Effect* (WBATE), and

$$\vartheta_{\texttt{LBATE}} = \tau_{\texttt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \tau(\mathbf{x})$$

is the *Largest Boundary Average Treatment Effect* (LBATE).

SA-2.3 Estimators and Other Related Quantities

The distance-based estimator process along the boundary is

$$\widehat{\vartheta}(\mathbf{x}) = \widehat{\theta}_{1,\mathbf{x}}(0) - \widehat{\theta}_{0,\mathbf{x}}(0), \qquad \mathbf{x}\in\mathscr{B},$$

where, for $t\in\{0,1\}$,

$$\widehat{\theta}_{t,\mathbf{x}}(0) = \mathbf{e}_0^\top \widehat{\boldsymbol{\gamma}}_t(\mathbf{x}), \qquad \widehat{\boldsymbol{\gamma}}_t(\mathbf{x}) = \underset{\boldsymbol{\gamma}\in\mathbb{R}^{p+1}}{\arg\min}\, \mathbb{E}_n\Big[\big(Y_i - \mathbf{r}_p(D_i(\mathbf{x})/h)^\top\boldsymbol{\gamma}\big)^2 K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\Big].$$

More generally, the least squares projection is

$$\widehat{\theta}_{t,\mathbf{x}}(r) = \mathbf{r}_p(r/h)^\top\widehat{\boldsymbol{\gamma}}_t(\mathbf{x}), \qquad t\in\{0,1\}, \quad \mathbf{x}\in\mathscr{B}.$$

The best mean square approximation is

$$\theta^*_{t,\mathbf{x}}(r) = \mathbf{r}_p(r/h)^\top\boldsymbol{\gamma}^*_t(\mathbf{x}),$$

where

$$\boldsymbol{\gamma}^*_t(\mathbf{x}) = \underset{\boldsymbol{\gamma}\in\mathbb{R}^{p+1}}{\arg\min}\, \mathbb{E}\Big[\big(Y_i - \mathbf{r}_p(D_i(\mathbf{x})/h)^\top\boldsymbol{\gamma}\big)^2 K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\Big],$$

and uniqueness will follow from the results below. Let

$$\theta^*_{t,\mathbf{x}}(0) = \mathbf{e}_0^\top\boldsymbol{\gamma}^*_t(\mathbf{x}).$$

The estimation error decomposes into *linear error*, *approximation error*, and *non-linear error*: for all $t\in\{0,1\}$ and $\mathbf{x}\in\mathscr{B}$,

$$\begin{aligned}
\widehat{\theta}_{t,\mathbf{x}}(0) - \theta_{t,\mathbf{x}}(0) &= \mathbf{e}_0^\top\widehat{\boldsymbol{\Psi}}^{-1}_{t,\mathbf{x}}\mathbb{E}_n\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)K_h(D_i(\mathbf{x}))Y_i\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\Big] - \theta_{t,\mathbf{x}}(0)\\
&= \mathbf{e}_0^\top\widehat{\boldsymbol{\Psi}}^{-1}_{t,\mathbf{x}}\mathbb{E}_n\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)K_h(D_i(\mathbf{x}))(Y_i - \theta^*_{t,\mathbf{x}}(D_i(\mathbf{x})))\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\Big]\\
&\quad + \theta^*_{t,\mathbf{x}}(0) - \theta_{t,\mathbf{x}}(0)\\
&= \underbrace{\theta^*_{t,\mathbf{x}}(0) - \theta_{t,\mathbf{x}}(0)}_{\text{approximation error}} + \underbrace{\mathbf{e}_0^\top\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}}_{\text{linear error}} + \underbrace{\mathbf{e}_0^\top(\widehat{\boldsymbol{\Psi}}^{-1}_{t,\mathbf{x}} - \boldsymbol{\Psi}^{-1}_{t,\mathbf{x}})\mathbf{O}_{t,\mathbf{x}}}_{\text{non-linear error}},
\end{aligned}$$

where

$$\mathbf{O}_{t,\mathbf{x}} = \mathbb{E}_n\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)K_h(D_i(\mathbf{x}))(Y_i - \theta^*_{t,\mathbf{x}}(D_i(\mathbf{x})))\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\Big],$$

with $\boldsymbol{\Psi}_{t,\mathbf{x}}$ defined above and

$$\widehat{\boldsymbol{\Psi}}_{t,\mathbf{x}} = \mathbb{E}_n\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)^\top K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t)\Big].$$

Therefore,

$$\widehat{\vartheta}(\mathbf{x}) - \vartheta(\mathbf{x}) = \mathfrak{B}(\mathbf{x}) + \mathfrak{L}(\mathbf{x}) + \mathfrak{Q}(\mathbf{x}). \tag{SA-1}$$

The misspecification bias is

$$\mathfrak{B}(\mathbf{x}) = \mathfrak{B}_1(\mathbf{x}) - \mathfrak{B}_0(\mathbf{x}), \qquad \mathfrak{B}_t(\mathbf{x}) = \theta^*_{t,\mathbf{x}}(0) - \theta_{t,\mathbf{x}}(0),$$

the linear term is

$$\mathfrak{L}(\mathbf{x}) = \mathbf{e}_0^\top \boldsymbol{\Psi}_{1,\mathbf{x}}^{-1}\mathbf{O}_{1,\mathbf{x}} - \mathbf{e}_0^\top \boldsymbol{\Psi}_{0,\mathbf{x}}^{-1}\mathbf{O}_{0,\mathbf{x}},$$

and the quadratic (non-linear) term is

$$\mathfrak{Q}(\mathbf{x}) = \mathbf{e}_0^\top(\widehat{\boldsymbol{\Psi}}_{1,\mathbf{x}}^{-1} - \boldsymbol{\Psi}_{1,\mathbf{x}}^{-1})\mathbf{O}_{1,\mathbf{x}} - \mathbf{e}_0^\top(\widehat{\boldsymbol{\Psi}}_{0,\mathbf{x}}^{-1} - \boldsymbol{\Psi}_{0,\mathbf{x}}^{-1})\mathbf{O}_{0,\mathbf{x}}.$$

The population projection defining $\theta^*_{t,\mathbf{x}}$ implies $\mathbb{E}[\mathfrak{L}(\mathbf{x})] = 0$ and $\mathbb{E}[\mathfrak{L}(\mathbf{x})^2] = \Xi_{\mathbf{x}} = \Xi_{\mathbf{x},\mathbf{x}}$, where

$$\Xi_{\mathbf{x}_1,\mathbf{x}_2} = \Xi_{0,\mathbf{x}_1,\mathbf{x}_2} + \Xi_{1,\mathbf{x}_1,\mathbf{x}_2}, \qquad \Xi_{t,\mathbf{x}_1,\mathbf{x}_2} = \frac{1}{nh^d}\mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}_1}^{-1}\boldsymbol{\Upsilon}_{t,\mathbf{x}_1,\mathbf{x}_2}\boldsymbol{\Psi}_{t,\mathbf{x}_2}^{-1}\mathbf{e}_0,$$

and

$$\begin{aligned}\boldsymbol{\Upsilon}_{t,\mathbf{x}_1,\mathbf{x}_2} = h^d\mathbb{E}\Big[&\mathbf{r}_p\Big(\frac{D_i(\mathbf{x}_1)}{h}\Big)\mathbf{r}_p\Big(\frac{D_i(\mathbf{x}_2)}{h}\Big)^\top K_h(D_i(\mathbf{x}_1))K_h(D_i(\mathbf{x}_2))\\ &(Y_i - \theta^*_{t,\mathbf{x}_1}(D_i(\mathbf{x}_1)))(Y_i - \theta^*_{t,\mathbf{x}_2}(D_i(\mathbf{x}_2)))\mathbb{1}(D_i(\mathbf{x}_1) \in \mathscr{I}_t)\mathbb{1}(D_i(\mathbf{x}_2) \in \mathscr{I}_t)\Big].\end{aligned}$$

Finally, for $t \in \{0,1\}$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{B}$, the sample analogues are $\widehat{\Xi}_{\mathbf{x}} = \widehat{\Xi}_{\mathbf{x},\mathbf{x}}$, where

$$\widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_2} = \widehat{\Xi}_{0,\mathbf{x}_1,\mathbf{x}_2} + \widehat{\Xi}_{1,\mathbf{x}_1,\mathbf{x}_2}, \qquad \widehat{\Xi}_{t,\mathbf{x}_1,\mathbf{x}_2} = \frac{1}{nh^d}\mathbf{e}_0^\top \widehat{\boldsymbol{\Psi}}_{t,\mathbf{x}_1}^{-1}\widehat{\boldsymbol{\Upsilon}}_{t,\mathbf{x}_1,\mathbf{x}_2}\widehat{\boldsymbol{\Psi}}_{t,\mathbf{x}_2}^{-1}\mathbf{e}_0,$$

and

$$\begin{aligned}\widehat{\boldsymbol{\Upsilon}}_{t,\mathbf{x}_1,\mathbf{x}_2} = h^d\mathbb{E}_n\Big[&\mathbf{r}_p\Big(\frac{D_i(\mathbf{x}_1)}{h}\Big)\mathbf{r}_p\Big(\frac{D_i(\mathbf{x}_2)}{h}\Big)^\top K_h(D_i(\mathbf{x}_1))K_h(D_i(\mathbf{x}_2))\\ &\cdot(Y_i - \widehat{\theta}_{t,\mathbf{x}_1}(D_i(\mathbf{x}_1)))(Y_i - \widehat{\theta}_{t,\mathbf{x}_2}(D_i(\mathbf{x}_2)))\mathbb{1}(D_i(\mathbf{x}_1) \in \mathscr{I}_t)\mathbb{1}(D_i(\mathbf{x}_2) \in \mathscr{I}_t)\Big].\end{aligned}$$

SA-2.4 Higher-Order Signed-Distance Derivatives

The main results in this appendix focus on the level parameter $\vartheta(\mathbf{x})$ because it equals the BATEC $\tau(\mathbf{x})$ under Theorem SA-1. One can also estimate derivatives of the induced signed-distance regression. For $\mathbf{x} \in \mathscr{B}$ and an integer $\nu \in \{1, \ldots, p\}$, whenever the one-sided limits exist, define

$$\theta^{(\nu)}_{t,\mathbf{x}}(0) = \lim_{r \to 0,\, r \in \mathscr{I}_t} \frac{d^\nu}{dr^\nu}\theta_{t,\mathbf{x}}(r), \qquad \vartheta^{(\nu)}(\mathbf{x}) = \theta^{(\nu)}_{1,\mathbf{x}}(0) - \theta^{(\nu)}_{0,\mathbf{x}}(0), \qquad \mathbf{x} \in \mathscr{B}.$$

The corresponding local-polynomial estimator replaces the level selector by the derivative selector:

$$\widehat{\vartheta}^{(\nu)}(\mathbf{x}) = \nu! h^{-\nu}\big(\mathbf{e}_\nu^\top \widehat{\boldsymbol{\gamma}}_1(\mathbf{x}) - \mathbf{e}_\nu^\top \widehat{\boldsymbol{\gamma}}_0(\mathbf{x})\big), \qquad \mathbf{x} \in \mathscr{B}.$$

All level results above have direct analogues after replacing $\mathbf{e}_0$ by $\nu! h^{-\nu}\mathbf{e}_\nu$, replacing the variance rate $(nh^d)^{-1}$ by $(nh^{d+2\nu})^{-1}$, and replacing the smooth-bias rate h^m by $h^{m-\nu}$ when the induced functions $r \mapsto \theta_{t,\mathbf{x}}(r)$ are m times differentiable. We do not state those analogues theorem by theorem because their notation is substantially heavier and the resulting parameter has a narrower causal interpretation: for $\nu > 0$, $\vartheta^{(\nu)}(\mathbf{x})$ is a derivative of an induced signed-distance regression, not a derivative of the BATEC curve as a function on $\mathscr{B}$.

There is, however, one useful first-derivative interpretation for $\mathbf{x} \in \mathscr{B}$ under a smooth Euclidean geometry. Suppose $\mathscr{d}$ is Euclidean, $\mathscr{B}$ is C^2 near $\mathbf{x}$, the treatment region has a well-defined unit normal $\mathbf{n}(\mathbf{x})$ pointing from control to treatment, meaning that $\mathbf{x} + r\mathbf{n}(\mathbf{x}) \in \mathscr{A}_1$ and $\mathbf{x} - r\mathbf{n}(\mathbf{x}) \in \mathscr{A}_0$ for all sufficiently small $r > 0$, and f_X, μ_0, and μ_1 are continuously differentiable near $\mathbf{x}$ with $f_X(\mathbf{x}) > 0$, so that the treatment-effect function τ admits the ambient gradient $\nabla\tau(\mathbf{x})$. Then,

$$\vartheta^{(1)}(\mathbf{x}) = c_d \, \nabla\tau(\mathbf{x})^\top \mathbf{n}(\mathbf{x}), \qquad c_d = \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d+1)/2)}, \qquad \mathbf{x} \in \mathscr{B}.$$

In the bivariate case, $c_2 = 2/\pi$. Thus, the first signed-distance derivative is proportional to the normal derivative of the treatment-effect surface. This formula uses the Euclidean metric and the local half-sphere geometry it induces; with a different metric, the relevant average is over the corresponding metric sphere and need not reduce to a Euclidean normal derivative.

SA-2.5 Preliminary Lemmas

Recall that $t \in \{0, 1\}$.

The following lemma gives a sufficient condition for Assumption SA-2(iii) when using the uniform kernel.

Lemma SA-1 (Primitive VC Condition for Distance Balls)**.** *Let $\mathscr{D}_\mathscr{d} = \{\{\mathbf{v} \in \mathscr{X} : \mathscr{d}(\mathbf{v}, \mathbf{x}) \le r\} : (\mathbf{x}, r) \in \mathscr{X} \times \mathbb{R}_+\}$. Suppose that, for some integer $J < \infty$, there are finite-dimensional linear spaces $\mathscr{V}_1, \ldots, \mathscr{V}_J$ of real-valued functions on $\mathscr{X}$ and a Boolean map $\mathsf{b} : \{0,1\}^J \to \{0,1\}$ such that, for every $(\mathbf{x}, r) \in \mathscr{X} \times \mathbb{R}_+$, there are $f_j(\cdot; \mathbf{x}, r) \in \mathscr{V}_j$, $j = 1, \ldots, J$, satisfying*

$$\mathbb{1}\{\mathscr{d}(\mathbf{v}, \mathbf{x}) \le r\} = \mathsf{b}\left(\mathbb{1}\{f_1(\mathbf{v}; \mathbf{x}, r) \ge 0\}, \ldots, \mathbb{1}\{f_J(\mathbf{v}; \mathbf{x}, r) \ge 0\}\right), \qquad \mathbf{v} \in \mathscr{X}.$$

Then, $\mathscr{D}_\mathscr{d}$ is a VC class.

In particular, the conclusion of Lemma SA-1 holds whenever the distance balls are semialgebraic sets of uniformly bounded degree and Boolean complexity. This includes the Euclidean distance, fixed Mahalanobis distances $\mathscr{d}(\mathbf{v}, \mathbf{x}) = ((\mathbf{v} - \mathbf{x})^\top A(\mathbf{v} - \mathbf{x}))^{1/2}$ with A positive definite, and distances generated by fixed polyhedral norms.

The following lemma gives a sufficient condition for Assumption SA-2(iv).

Lemma SA-2 (Gram Invertibility)**.** *Suppose the following conditions hold:*

(a) *Assumptions SA-1(i)(ii) and Assumption SA-2(iii) hold.*
(b) *$\mathscr{d}(\cdot, \cdot)$ is the Euclidean distance.*

(c) *There exists a compact set* $U \subseteq \mathbb{R}^d$, *such that* $K(\varsigma_t\|\mathbf{u}\|) > 0$ *for all* $\mathbf{u} \in U$, $\lambda_{\min}(\int_U \mathbf{r}_p(\|\mathbf{z}\|)\mathbf{r}_p(\|\mathbf{z}\|)^\top \,\mathrm{d}\mathbf{z}) > 0$, *and* $\liminf_{h\downarrow 0} \inf_{\mathbf{x}\in\mathscr{B}} \int_U K(\varsigma_t\|\mathbf{u}\|)\mathbb{1}(\mathbf{x}+h\mathbf{u}\in\mathscr{A}_t)\,\mathrm{d}\mathbf{u} \gtrsim 1$.

Then, Assumption SA-2(iv) holds.

Lemma SA-3 (Gram). *Suppose Assumptions SA-1(i)(ii) and SA-2 hold. If* $\frac{nh^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$, *then*

$$\sup_{\mathbf{x}\in\mathscr{B}} \big\|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}} - \mathbf{\Psi}_{t,\mathbf{x}}\big\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}}, \qquad 1 \lesssim_{\mathbb{P}} \inf_{\mathbf{x}\in\mathscr{B}} \lambda_{\min}\big(\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}\big) \le \sup_{\mathbf{x}\in\mathscr{B}} \big\|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}\big\| \lesssim_{\mathbb{P}} 1,$$
$$\sup_{\mathbf{x}\in\mathscr{B}} \big\|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1} - \mathbf{\Psi}_{t,\mathbf{x}}^{-1}\big\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}}.$$

Lemma SA-4 (Stochastic Linear Approximation). *Suppose Assumptions SA-1(i)(ii)(iii)(v) and SA-2 hold. If* $\frac{nh^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$, *then*

$$\sup_{\mathbf{x}\in\mathscr{B}} \big\|\mathbf{O}_{t,\mathbf{x}}\big\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d},$$
$$\sup_{\mathbf{x}\in\mathscr{B}} \big|\mathbf{e}_0^\top \mathbf{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{O}_{t,\mathbf{x}}\big| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d},$$
$$\sup_{\mathbf{x}\in\mathscr{B}} \big|\mathbf{e}_0^\top (\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1} - \mathbf{\Psi}_{t,\mathbf{x}}^{-1})\mathbf{O}_{t,\mathbf{x}}\big| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}}\left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d}\right).$$

Lemma SA-5 (Covariance). *Suppose Assumptions SA-1 and SA-2 hold. If* $\frac{nh^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$, *then*

$$1 \lesssim \inf_{\mathbf{x}\in\mathscr{B}} nh^d \Xi_{t,\mathbf{x},\mathbf{x}} \le \sup_{\mathbf{x}\in\mathscr{B}} nh^d \Xi_{t,\mathbf{x},\mathbf{x}} \lesssim 1,$$

$$\sup_{\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}} \big\|\widehat{\mathbf{\Upsilon}}_{t,\mathbf{x}_1,\mathbf{x}_2} - \mathbf{\Upsilon}_{t,\mathbf{x}_1,\mathbf{x}_2}\big\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d},$$

and

$$\sup_{\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}} nh^d \big|\widehat{\Xi}_{t,\mathbf{x}_1,\mathbf{x}_2} - \Xi_{t,\mathbf{x}_1,\mathbf{x}_2}\big| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}.$$

If, in addition, $\frac{n^{\frac{v}{2+v}}h^d}{\log(1/h)} \to \infty$, *then*

$$\inf_{\mathbf{x}\in\mathscr{B}} \lambda_{\min}(\widehat{\mathbf{\Upsilon}}_{t,\mathbf{x},\mathbf{x}}) \gtrsim_{\mathbb{P}} 1, \qquad \inf_{\mathbf{x}\in\mathscr{B}} nh^d \widehat{\Xi}_{t,\mathbf{x},\mathbf{x}} \gtrsim_{\mathbb{P}} 1, \qquad \sup_{\mathbf{x}\in\mathscr{B}} nh^d \widehat{\Xi}_{t,\mathbf{x},\mathbf{x}} \lesssim_{\mathbb{P}} 1,$$

and

$$\sup_{\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}} \left|\frac{\widehat{\Xi}_{t,\mathbf{x}_1,\mathbf{x}_2}}{\sqrt{\widehat{\Xi}_{t,\mathbf{x}_1,\mathbf{x}_1}\widehat{\Xi}_{t,\mathbf{x}_2,\mathbf{x}_2}}} - \frac{\Xi_{t,\mathbf{x}_1,\mathbf{x}_2}}{\sqrt{\Xi_{t,\mathbf{x}_1,\mathbf{x}_1}\Xi_{t,\mathbf{x}_2,\mathbf{x}_2}}}\right| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}.$$

SA-3 Boundary Average Treatment Effect Curve

SA-3.1 Identification

Theorem SA-1 (Distance-Based Identification)**.** *Suppose Assumptions SA-1(i)-(iii) and SA-2(i), (ii), and (v) hold. Then,*

$$\tau(\mathbf{x}) = \vartheta(\mathbf{x}) = \lim_{r\downarrow 0} \theta_{1,\mathbf{x}}(r) - \lim_{r\uparrow 0} \theta_{0,\mathbf{x}}(r), \qquad \text{for all } \mathbf{x} \in \mathscr{B}.$$

SA-3.2 Approximation Bias

The next two theorems collect the approximation-bias results corresponding to Theorems 2 and 3 in the paper. The first theorem gives a dimension-generic level upper bound and the bivariate sharpness result; the second theorem gives improved level bias bounds under smoothness of the induced signed-distance regression.

Theorem SA-2 (Approximation Bias: Minimal Level Guarantee)**.** *Suppose Assumptions SA-1(i)–(iii) and SA-2 hold. If $h = o(1)$, then*

$$\sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}(\mathbf{x})| \lesssim h,$$

for any $d \geq 1$. In particular, for $d = 2$, this bias-rate is sharp in the following sense. For some $L \geq 1$, let $\mathscr{P}$ be the class of data generating processes satisfying Assumptions SA-1(i)–(iii) with $\mathscr{X} \subseteq [-L, L]^2$, Assumption SA-2, and the following additional conditions, with (ii) and (iii) imposed for each $t \in \{0, 1\}$:

(i) $L^{-1} \leq \inf_{\mathbf{x}\in\mathscr{X}} f_X(\mathbf{x}) \leq \sup_{\mathbf{x}\in\mathscr{X}} f_X(\mathbf{x}) \leq L$,
(ii) $\max_{0\leq|\boldsymbol{\nu}|\leq p} \sup_{\mathbf{x}\in\mathscr{X}} |\partial^{\boldsymbol{\nu}}\mu_t(\mathbf{x})| + \max_{0\leq|\boldsymbol{\nu}|\leq p} \sup_{\mathbf{x},\mathbf{y}\in\mathscr{X}} \frac{|\partial^{\boldsymbol{\nu}}\mu_t(\mathbf{x}) - \partial^{\boldsymbol{\nu}}\mu_t(\mathbf{y})|}{\|\mathbf{x}-\mathbf{y}\|} \leq L$, *and*
(iii) $\liminf_{h\downarrow 0} \inf_{\mathbf{x}\in\mathscr{B}} \int_{\mathscr{A}_t} K_h((2t-1)\mathscr{d}(\mathbf{u}, \mathbf{x}))\, \mathrm{d}\mathbf{u} \geq L^{-1}$.

If $nh^2 \to \infty$ and $h = o(1)$, then

$$1 \lesssim \liminf_{n\to\infty} \sup_{\mathbb{P}\in\mathscr{P}} \sup_{\mathbf{x}\in\mathscr{B}} \frac{|\mathfrak{B}(\mathbf{x})|}{h} \leq \limsup_{n\to\infty} \sup_{\mathbb{P}\in\mathscr{P}} \sup_{\mathbf{x}\in\mathscr{B}} \frac{|\mathfrak{B}(\mathbf{x})|}{h} \lesssim 1.$$

The first part of Theorem SA-2 is dimension-generic. The sharpness statement is a level result: near kinked boundaries, signed-distance derivatives need not exist without additional smoothness restrictions.

Theorem SA-3 (Approximation Bias: Smooth Boundary)**.** *Suppose Assumptions SA-1(i)–(iii) and SA-2 hold, and let $h = o(1)$.*

(i) *If, for some integer $m \in \{1, \ldots, p+1\}$, each map $r \mapsto \theta_{t,\mathbf{x}}(r)$ is m-times continuously differentiable on $\mathscr{I}_t$ near zero with mth derivative uniformly bounded over $t \in \{0, 1\}$ and $\mathbf{x} \in \mathscr{B}$, then*

$$\sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}(\mathbf{x})| \lesssim h^m.$$

The pointwise analogue holds under the same smoothness condition at a fixed $\mathbf{x} \in \mathscr{B}$.

(ii) *Suppose $d = 2$ and $\mathscr{d}(\cdot,\cdot)$ is the Euclidean distance. For $\mathbf{x} \in \mathscr{B}$, suppose that for some $\delta, \varepsilon > 0$, $\mathscr{B} \cap \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| \leq \varepsilon\} = \gamma([-\delta, \delta])$, where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is one-to-one and belongs to $C^{\kappa+2}([-\delta, \delta], \mathbb{R}^2)$.*

Let $m_{\mathbf{x}} = \kappa \wedge (p+1)$. Then, $\theta_{0,\mathbf{x}}(\cdot)$ and $\theta_{1,\mathbf{x}}(\cdot)$ are $m_{\mathbf{x}}$-times continuously differentiable near zero, and

$$|\mathfrak{B}(\mathbf{x})| \lesssim h^{m_{\mathbf{x}}}.$$

(iii) *Suppose $d = 2$, $\mathscr{d}(\cdot,\cdot)$ is the Euclidean distance, and $\mathscr{B} = \gamma([0,L])$, where γ is one-to-one and belongs to $C^{\iota+2}([0,L],\mathbb{R}^2)$ for some $L > 0$. Suppose there exist $\delta, \varepsilon > 0$ such that, for all $\mathbf{x} \in \mathscr{B}^o = \gamma([\delta, L-\delta])$, $r \in [0,\varepsilon]$, and $t \in \{0,1\}$, the set $\{\mathbf{u} \in \mathbb{R}^2 : \mathscr{d}(\mathbf{u},\mathbf{x}) = r\}$ intersects $\mathrm{bd}(\mathscr{A}_t)$ at only two points in $\mathscr{B}$. Let $m = \iota \wedge (p+1)$. Then, $\theta_{0,\mathbf{x}}(\cdot)$ and $\theta_{1,\mathbf{x}}(\cdot)$ are m-times continuously differentiable near zero with derivatives uniformly bounded over $\mathbf{x} \in \mathscr{B}^o$, and*

$$\sup_{\mathbf{x}\in\mathscr{B}^o} |\mathfrak{B}(\mathbf{x})| \lesssim h^m.$$

SA-3.3 Point Estimation and Approximate MSE Expansions

Theorem SA-4 (Convergence Rates). *Suppose Assumptions SA-1 and SA-2 hold. Let $h = o(1)$.*

- *If $\vartheta(\mathbf{x})$ is well defined at a fixed $\mathbf{x} \in \mathscr{B}$, and $nh^d \to \infty$, then*

$$\left|\widehat{\vartheta}(\mathbf{x}) - \vartheta(\mathbf{x})\right| \lesssim_{\mathbb{P}} \frac{1}{\sqrt{nh^d}} + \frac{1}{n^{\frac{1+v}{2+v}} h^d} + \left|\mathfrak{B}(\mathbf{x})\right|.$$

- *If $\vartheta(\mathbf{x})$ is well defined for all $\mathbf{x} \in \mathscr{B}$, and $\frac{nh^d}{\log(1/h)} \to \infty$, then*

$$\sup_{\mathbf{x}\in\mathscr{B}} \left|\widehat{\vartheta}(\mathbf{x}) - \vartheta(\mathbf{x})\right| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}} h^d} + \sup_{\mathbf{x}\in\mathscr{B}} \left|\mathfrak{B}(\mathbf{x})\right|.$$

Equation (SA-1) also gives MSE-type criteria for bandwidth choice. Unlike the location-based estimator, the distance-based estimator is based on a one-dimensional projection of the full score, so a conditional-on-$\mathbf{X}$ MSE does not isolate a clean local-polynomial variance term under misspecification. Instead, we use the linearized decomposition implied by (SA-1). Because the population projection defining $\theta^*_{t,\mathbf{x}}$ implies $\mathbb{E}[\mathfrak{L}(\mathbf{x})] = 0$ and $\mathbb{E}[\mathfrak{L}(\mathbf{x})^2] = \Xi_{\mathbf{x}}$, we define the approximate MSE and approximate IMSE as

$$\mathrm{AMSE}(\mathbf{x}) = \mathbb{E}\big[(\mathfrak{B}(\mathbf{x}) + \mathfrak{L}(\mathbf{x}))^2\big] = \mathfrak{B}(\mathbf{x})^2 + \Xi_{\mathbf{x}}, \qquad \mathbf{x} \in \mathscr{B},$$

and the approximate IMSE

$$\mathrm{AIMSE} = \int_{\mathscr{B}} \mathrm{AMSE}(\mathbf{x}) w(\mathbf{x}) \, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).$$

The results in Section SA-3.2 give different approximation and exact rate results for the bias $\mathfrak{B}(\mathbf{x})$, Lemma SA-5 implies that $V_{\mathbf{x}} = nh^d \Xi_{\mathbf{x}}$ is bounded and bounded away from zero, and Lemma SA-4 implies that

$$\sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{Q}(\mathbf{x})| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} \left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}} h^d} \right).$$

Therefore, under appropriate rate conditions, the nonlinear term is higher order and can be ignored for first-order bandwidth calculations.

For bandwidth selection, suppose that a leading-bias expansion $\mathfrak{B}(\mathbf{x}) = h^m B_{\mathbf{x}} + o(h^m)$ is available, with $B_{\mathbf{x}}$ bounded and bounded away from zero. The approximate MSE criterion is then

$$h^{2m} B_{\mathbf{x}}^2 + \frac{1}{nh^d} V_{\mathbf{x}},$$

which gives the approximate MSE-optimal bandwidth selection rule

$$h_{\mathtt{AMSE},\mathbf{x}} = \Big(\frac{dV_{\mathbf{x}}}{2mB_{\mathbf{x}}^2} \frac{1}{n} \Big)^{\frac{1}{2m+d}}.$$

Similarly, if $\int_{\mathscr{B}} B_{\mathbf{x}}^2 w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})$ is bounded and bounded away from zero, the approximate IMSE-optimal bandwidth choice is

$$h_{\mathtt{AIMSE}} = \Big(\frac{d \int_{\mathscr{B}} V_{\mathbf{x}} w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})}{2m \int_{\mathscr{B}} B_{\mathbf{x}}^2 w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})} \frac{1}{n} \Big)^{\frac{1}{2m+d}}.$$

Here m is the order of the signed-distance approximation bias. Theorems SA-2 and SA-3 give primitive conditions under which such rates are available. In contrast to the location-based case in Cattaneo et al. [2026], this bias order is governed by the smoothness of the induced signed-distance regression, and therefore by the geometry of the boundary and the distance metric, not only by the polynomial order p.

SA-3.4 Distributional Approximation and Inference

Let $\mathbf{V} = ((\mathbf{X}_1^\top, Y_1), \cdots, (\mathbf{X}_n^\top, Y_n))$, and recall that $t \in \{0, 1\}$. The feasible t-statistic is

$$\widehat{\mathrm{T}}(\mathbf{x}) = \frac{\widehat{\vartheta}(\mathbf{x}) - \vartheta(\mathbf{x})}{\sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}}}, \qquad \mathbf{x} \in \mathscr{B}.$$

The associated $100(1-\alpha)\%$ confidence interval estimator is

$$\widehat{\mathrm{I}}_\alpha(\mathbf{x}) = \Big[\widehat{\vartheta}(\mathbf{x}) - \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}}\ ,\ \widehat{\vartheta}(\mathbf{x}) + \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}} \Big],$$

where $\mathfrak{q}_\alpha$ denotes an appropriate quantile depending on the desired confidence level $\alpha \in (0,1)$ and coverage objective (pointwise vs. uniform over $\mathscr{B}$). The following theorem establishes pointwise asymptotic normality and validity of confidence intervals. Let $\Phi(\cdot)$ be the cumulative distribution function of a standard univariate Gaussian random variable.

Theorem SA-5 (Confidence Intervals)**.** *Suppose Assumptions SA-1 and SA-2 hold, and $\vartheta(\mathbf{x})$ is well defined. If $n^{\frac{v}{2+v}} h^d \to \infty$, $h = o(1)$, and $\sqrt{nh^d}|\mathfrak{B}(\mathbf{x})| \to 0$, then*

$$\sup_{u \in \mathbb{R}} \Big| \mathbb{P}\big(\widehat{\mathrm{T}}(\mathbf{x}) \le u\big) - \Phi(u) \Big| = o(1), \qquad \mathbf{x} \in \mathscr{B},$$

and

$$\mathbb{P}\big(\vartheta(\mathbf{x}) \in \widehat{\mathrm{I}}_\alpha(\mathbf{x})\big) = 1 - \alpha + o(1), \qquad \mathbf{x} \in \mathscr{B},$$

provided that $\mathfrak{q}_\alpha = \inf\{c > 0 : \mathbb{P}(|\widehat{Z}| \ge c \mid \mathbf{V}) \le \alpha\}$ with $\widehat{Z} \mid \mathbf{V} \sim \mathsf{Normal}(0,1)$.

To conduct uniform inference, we rely on a new strong approximation result established in Section SA-7. First, we approximate (uniformly over $\mathbf{x} \in \mathscr{B}$) the feasible statistic $\widehat{\mathrm{T}}$ by the following linear statistic (which is a sum of independent random variables):

$$\overline{\mathrm{T}}(\mathbf{x}) = \big(\Xi_{\mathbf{x},\mathbf{x}}\big)^{-1/2}\Big(\mathbf{e}_0^\top \boldsymbol{\Psi}_{1,\mathbf{x}}^{-1}\mathbf{O}_{1,\mathbf{x}} - \mathbf{e}_0^\top \boldsymbol{\Psi}_{0,\mathbf{x}}^{-1}\mathbf{O}_{0,\mathbf{x}}\Big), \qquad \mathbf{x} \in \mathscr{B}.$$

Theorem SA-6 (Stochastic Linearization)**.** *Suppose Assumptions SA-1 and SA-2 hold, and $\vartheta(\mathbf{x})$ is well defined for all $\mathbf{x} \in \mathscr{B}$. If $\frac{n^{\frac{v}{2+v}}h^d}{\log(1/h)} \to \infty$ and $h = o(1)$, then*

$$\sup_{\mathbf{x}\in\mathscr{B}}\big|\widehat{\mathrm{T}}(\mathbf{x}) - \overline{\mathrm{T}}(\mathbf{x})\big| \lesssim_{\mathbb{P}} \sqrt{\log(1/h)}\left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}\right) + \sqrt{nh^d}\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}(\mathbf{x})|.$$

The pointwise (in $\mathscr{B}$) analogue of this result removes the $\log(1/h)$ penalty. See the proof of Theorem SA-5 for more details. To establish a Gaussian strong approximation for $\overline{\mathrm{T}}(\mathbf{x})$, define the class of functions $\mathscr{G} = \{g_{\mathbf{x}} : \mathbf{x} \in \mathscr{B}\}$ and $\mathscr{M} = \{m_{\mathbf{x}} : \mathbf{x} \in \mathscr{B}\}$, where

$$\begin{aligned} g_{\mathbf{x}}(\mathbf{u}) &= \mathbb{1}(\mathbf{u} \in \mathscr{A}_1)\mathfrak{K}_1(\mathbf{u};\mathbf{x}) - \mathbb{1}(\mathbf{u} \in \mathscr{A}_0)\mathfrak{K}_0(\mathbf{u};\mathbf{x}), \\ m_{\mathbf{x}}(\mathbf{u}) &= -\mathbb{1}(\mathbf{u} \in \mathscr{A}_1)\mathfrak{K}_1(\mathbf{u};\mathbf{x})\theta^*_{1,\mathbf{x}}(D_{\mathbf{u}}(\mathbf{x})) + \mathbb{1}(\mathbf{u} \in \mathscr{A}_0)\mathfrak{K}_0(\mathbf{u};\mathbf{x})\theta^*_{0,\mathbf{x}}(D_{\mathbf{u}}(\mathbf{x})), \end{aligned} \tag{SA-2}$$

with

$$\mathfrak{K}_t(\mathbf{u};\mathbf{x}) = \frac{1}{\sqrt{n\Xi_{\mathbf{x},\mathbf{x}}}}\mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{r}_p\Big(\frac{D_{\mathbf{u}}(\mathbf{x})}{h}\Big)K_h(D_{\mathbf{u}}(\mathbf{x})),$$

for all $\mathbf{u} \in \mathscr{X}$, $\mathbf{x} \in \mathscr{B}$, and $t \in \{0,1\}$. In addition, let $\mathscr{R}$ be the class of functions containing the singleton identity function $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$, $\mathrm{Id}(x) = x$. Then, $\overline{\mathrm{T}}(\mathbf{x})$ can be represented as

$$\overline{\mathrm{T}}(\mathbf{x}) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \Big[g_{\mathbf{x}}(\mathbf{X}_i)\,\mathrm{Id}(Y_i) + m_{\mathbf{x}}(\mathbf{X}_i) - \mathbb{E}\big[g_{\mathbf{x}}(\mathbf{X}_i)\,\mathrm{Id}(Y_i) + m_{\mathbf{x}}(\mathbf{X}_i)\big]\Big].$$

Define the multiplicative-separable empirical process as

$$M_n(g,r) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \big[g(\mathbf{x}_i)r(y_i) - \mathbb{E}[g(\mathbf{x}_i)r(y_i)]\big], \qquad g \in \mathscr{G}, r \in \mathscr{R},$$

which implies that

$$\overline{\mathrm{T}}(\mathbf{x}) = M_n(g_{\mathbf{x}}, \mathrm{Id}) + M_n(m_{\mathbf{x}}, 1), \qquad \mathbf{x} \in \mathscr{B}.$$

Theorem SA-25 gives a new Gaussian strong approximation that can be applied to $\overline{\mathrm{T}}(\mathbf{x})$. This new theorem allows polynomial moment bounds on the conditional distribution of $Y_i|\mathbf{X}_i$.

Theorem SA-7 (Gaussian Strong Approximation: $\overline{\mathrm{T}}$)**.** *Suppose Assumptions SA-1 and SA-2 hold, and $\mathscr{X} = \prod_{l=1}^d [a_l, b_l]$, with $-\infty < a_l < b_l < \infty$ for $l = 1, \ldots, d$. Suppose also that there exists a constant $C > 0$ such that for $t \in \{0,1\}$ and for any $\mathbf{x} \in \mathscr{B}$, the De Giorgi perimeter of the localized set $E_{t,\mathbf{x}}(h) = \{\mathbf{y} \in \mathscr{A}_t : \varsigma_t \mathscr{d}(\mathbf{y},\mathbf{x})/h \in \mathrm{supp}(K)\}$ satisfies $\mathscr{L}(E_{t,\mathbf{x}}(h)) \leq Ch^{d-1}$. If $\liminf_{n\to\infty}\frac{\log h}{\log n} > -\infty$, $nh^d \to \infty$, and $h = o(1)$, then (on a possibly enlarged probability space) there exists a mean-zero Gaussian process Z indexed by $\mathscr{B}$*

with almost surely continuous sample path such that

$$\mathbb{E}\Big[\sup_{\mathbf{x}\in\mathscr{B}}\big|\overline{\mathrm{T}}(\mathbf{x})-Z(\mathbf{x})\big|\Big]\lesssim(\log(n))^{\frac{3}{2}}\Big(\frac{1}{nh^d}\Big)^{\frac{1}{2d+2}\frac{v}{v+2}}+\log(n)\Big(\frac{1}{n^{\frac{v}{2+v}}h^d}\Big)^{\frac{1}{2}},$$

where $\lesssim$ *is up to a universal constant, and* Z *has the same covariance structure as* $\overline{\mathrm{T}}$*; that is,* $\mathbb{C}\mathrm{ov}[\overline{\mathrm{T}}(\mathbf{x}_1),\overline{\mathrm{T}}(\mathbf{x}_2)]=\mathbb{C}\mathrm{ov}[Z(\mathbf{x}_1),Z(\mathbf{x}_2)]$ *for all* $\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}$.

The rectangular-support condition in Theorem SA-7 is used only to invoke the normalizing-map construction in Cattaneo and Yu [2025, Section 3.1]. After a componentwise affine rescaling, one can take $\mathscr{X}=[0,1]^d$; because f_X is continuous and bounded away from zero on this compact rectangle, $\mathbb{P}_X$ is a valid surrogate measure and the Rosenblatt transform provides a valid normalizing transformation with finite constants. The De Giorgi perimeter condition controls the total variation of the localized distance-kernel classes.

Theorem SA-7 can be used to construct confidence bands for $(\vartheta(\mathbf{x}):\mathbf{x}\in\mathscr{B})$. Let $(\widehat{Z}(\mathbf{x}):\mathbf{x}\in\mathscr{B})$ be a (conditionally on $\mathbf{V}$) mean-zero Gaussian process with feasible (conditional) covariance function

$$\mathbb{C}\mathrm{ov}\Big[\widehat{Z}(\mathbf{x}_1),\widehat{Z}(\mathbf{x}_2)\Big|\mathbf{V}\Big]=\frac{\widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_2}}{\sqrt{\widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_1}}\sqrt{\widehat{\Xi}_{\mathbf{x}_2,\mathbf{x}_2}}},\qquad\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}.$$

Theorem SA-8 (Confidence Bands)**.** *Suppose the assumptions and conditions in Theorem SA-7 hold, and* $\vartheta(\mathbf{x})$ *is well defined for all* $\mathbf{x}\in\mathscr{B}$*. If* $\liminf_{n\to\infty}\frac{\log h}{\log n}>-\infty$*,* $\frac{n^{\frac{v}{2+v}}h^d}{(\log n)^3}\to\infty$*, and* $\sqrt{nh^d\log n}\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}(\mathbf{x})|\to 0$*, then*

$$\sup_{u\in\mathbb{R}}\Big|\mathbb{P}\Big(\sup_{\mathbf{x}\in\mathscr{B}}\big|\widehat{\mathrm{T}}(\mathbf{x})\big|\le u\Big)-\mathbb{P}\Big(\sup_{\mathbf{x}\in\mathscr{B}}\big|\widehat{Z}(\mathbf{x})\big|\le u\Big|\mathbf{V}\Big)\Big|=o_{\mathbb{P}}(1)$$

and

$$\mathbb{P}\Big[\vartheta(\mathbf{x})\in\widehat{\mathrm{I}}_\alpha(\mathbf{x}),\ \textit{for all}\ \mathbf{x}\in\mathscr{B}\Big]=1-\alpha+o(1),$$

provided that $\mathfrak{q}_\alpha=\inf\big\{c>0:\mathbb{P}\big(\sup_{\mathbf{x}\in\mathscr{B}}\big|\widehat{Z}(\mathbf{x})\big|\ge c\big|\mathbf{V}\big)\le\alpha\big\}$.

Theorem SA-7 gives a strong approximation of the entire process $\overline{\mathrm{T}}$ by a Gaussian process. This process-level coupling is stronger than what is needed to construct confidence bands. The following theorem approximates only the supremum of the statistic. It avoids the rectangular-support and De Giorgi perimeter conditions used in the process-level result.

Theorem SA-9 (Gaussian Approximation for $\sup_{\mathbf{x}\in\mathscr{B}}|\overline{\mathrm{T}}(\mathbf{x})|$)**.** *Suppose Assumptions SA-1 and SA-2 hold,* $\liminf_{n\to\infty}\frac{\log h}{\log n}>-\infty$*, and* $h=o(1)$*. Then, for* n *large enough, on a possibly enlarged probability space there exists a mean-zero Gaussian process* Z *indexed by* $\mathscr{B}$ *with almost surely continuous sample paths such that*

$$\mathbb{C}\mathrm{ov}[Z(\mathbf{x}_1),Z(\mathbf{x}_2)]=\mathbb{C}\mathrm{ov}[\overline{\mathrm{T}}(\mathbf{x}_1),\overline{\mathrm{T}}(\mathbf{x}_2)],\qquad\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B},$$

and

$$\mathbb{P}\left[\Big|\sup_{\mathbf{x}\in\mathscr{B}}|\overline{\mathrm{T}}(\mathbf{x})|-\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|\Big|>\frac{\log n}{(nh^d)^{1/6}}+\frac{(\log n)^{5/4}}{(nh^d)^{1/4}}+\frac{(\log n)^{3/2}}{(n^{\frac{v}{2+v}}h^d)^{1/2}}\right]\lesssim\frac{1}{\log n}+\frac{\log n}{n}.$$

Using Theorem SA-9, we obtain the following confidence-band result. Relative to Theorem SA-8, it removes the process-level geometric side conditions at the cost of a stronger logarithmic rate condition.

Theorem SA-10 (Confidence Bands: Suprema Approximation)**.** *Suppose Assumptions SA-1 and SA-2 hold, and $\vartheta(\mathbf{x})$ is well defined for all $\mathbf{x} \in \mathscr{B}$. If $\liminf_{n\to\infty} \frac{\log h}{\log n} > -\infty$, $\frac{n^{\frac{v}{2+v}} h^d}{(\log n)^9} \to \infty$, $h = o(1)$, and $\sqrt{nh^d \log n}\sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}(\mathbf{x})| \to 0$, then*

$$\sup_{u\in\mathbb{R}} \left| \mathbb{P}\Big(\sup_{\mathbf{x}\in\mathscr{B}} \left|\widehat{\mathrm{T}}(\mathbf{x})\right| \le u \Big) - \mathbb{P}\Big(\sup_{\mathbf{x}\in\mathscr{B}} \left|\widehat{Z}(\mathbf{x})\right| \le u \Big| \mathbf{V} \Big) \right| = o_{\mathbb{P}}(1)$$

and

$$\mathbb{P}\Big[\vartheta(\mathbf{x}) \in \widehat{\mathrm{I}}_\alpha(\mathbf{x}),\ \textit{for all } \mathbf{x} \in \mathscr{B} \Big] = 1 - \alpha + o(1),$$

provided that $\mathfrak{q}_\alpha = \inf\big\{ c > 0 : \mathbb{P}\big(\sup_{\mathbf{x}\in\mathscr{B}} \left|\widehat{Z}(\mathbf{x})\right| \ge c \big| \mathbf{V} \big) \le \alpha \big\}$.

SA-4 Weighted Boundary Average Treatment Effect

Without loss of generality, normalize the weight function so that $\int_{\mathscr{B}} w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}) = 1$. Define the distance-based WBATE parameter and plug-in estimator by

$$\vartheta_{\mathtt{WBATE}} = \int_{\mathscr{B}} \vartheta(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}), \qquad \text{and} \qquad \widehat{\vartheta}_{\mathtt{WBATE}} = \int_{\mathscr{B}} \widehat{\vartheta}(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).$$

Integrating the decomposition (SA-1) for the difference of the two side-specific estimators gives

$$\widehat{\vartheta}_{\mathtt{WBATE}} - \vartheta_{\mathtt{WBATE}} = \mathfrak{B}_{\mathtt{WBATE}} + \mathfrak{L}_{\mathtt{WBATE}} + \mathfrak{Q}_{\mathtt{WBATE}}, \tag{SA-3}$$

where

$$\mathfrak{B}_{\mathtt{WBATE}} = \int_{\mathscr{B}} \mathfrak{B}(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}), \qquad \mathfrak{L}_{\mathtt{WBATE}} = \int_{\mathscr{B}} \mathfrak{L}(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}),$$

and

$$\mathfrak{Q}_{\mathtt{WBATE}} = \int_{\mathscr{B}} \mathfrak{Q}(\mathbf{x}) w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).$$

Define the aggregate variance of the leading term by

$$\Xi_{\mathtt{WBATE}} = \int_{\mathscr{B}} \int_{\mathscr{B}} \Xi_{\mathbf{x}_1,\mathbf{x}_2} w(\mathbf{x}_1) w(\mathbf{x}_2)\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_1)\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_2).$$

Its feasible counterpart is

$$\widehat{\Xi}_{\mathtt{WBATE}} = \int_{\mathscr{B}} \int_{\mathscr{B}} \widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_2} w(\mathbf{x}_1) w(\mathbf{x}_2)\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_1)\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_2).$$

For the aggregate stochastic rate and inference results, impose the local boundary-measure regularity con-

dition

$$\sup_{\mathbf{x}\in\mathscr{B}} \mathfrak{H}^{d-1}(\mathscr{B}\cap B(\mathbf{x},r)) \leq C_{\mathscr{B}} r^{d-1}, \qquad 0<r\leq r_0, \tag{SA-4}$$

for some constants $C_{\mathscr{B}}<\infty$ and $r_0>0$, where $B(\mathbf{x},r)$ denotes a Euclidean ball. This upper local regularity condition holds, for example, for compact piecewise C^1 boundaries with finite local multiplicity, including piecewise-linear boundaries. Write $\nu_{\mathscr{B},k}$ for the k-fold product measure $(\mathfrak{H}^{d-1})^k$ on $\mathscr{B}^k$. For inference, impose the scalar aggregate nondegeneracy condition

$$\Xi_{\texttt{WBATE}} \gtrsim \frac{h^{d-1}}{nh^d} = \frac{1}{nh}, \tag{SA-5}$$

which rules out cancellation of the integrated covariance kernel.

Lemma SA-6 (Variance Linear Term: WBATE)**.** *Suppose Assumptions SA-1, SA-2, and SA-3 hold, and (SA-4) holds. If* $\frac{nh^d}{\log(1/h)}\to\infty$ *and* $h=o(1)$*, then*

$$\mathbb{V}[\mathfrak{L}_{\texttt{WBATE}}] = \Xi_{\texttt{WBATE}} \lesssim \frac{1}{nh}.$$

If, in addition, (SA-5) holds, then

$$\frac{1}{nh} \lesssim \Xi_{\texttt{WBATE}} \lesssim \frac{1}{nh}.$$

If $\frac{n^{v/(2+v)}h^d}{\log(1/h)}\to\infty$ *and* $h=o(1)$*, then*

$$\widehat{\Xi}_{\texttt{WBATE}} = \Xi_{\texttt{WBATE}} + o_{\mathbb{P}}\Big(\frac{1}{nh}\Big).$$

If, in addition, (SA-5) holds, then $\widehat{\Xi}_{\texttt{WBATE}}/\Xi_{\texttt{WBATE}} \to_{\mathbb{P}} 1$.

Lemma SA-7 (Quadratic Remainder: WBATE)**.** *Suppose Assumptions SA-1, SA-2, and SA-3 hold, and (SA-4) holds. If* $\frac{nh^d}{\log(1/h)}\to\infty$ *and* $h=o(1)$*, then*

$$\mathfrak{Q}_{\texttt{WBATE}} \lesssim_{\mathbb{P}} \frac{1}{nh^d}.$$

If, in addition, $nh^{2d-1}\to\infty$*, then*

$$\mathfrak{Q}_{\texttt{WBATE}} = o_{\mathbb{P}}\Big(\frac{1}{\sqrt{nh}}\Big).$$

Theorem SA-11 (Convergence Rate: WBATE)**.** *Suppose Assumptions SA-1, SA-2, and SA-3 hold,* $\vartheta(\mathbf{x})$ *is well defined for all* $\mathbf{x}\in\mathscr{B}$*, and (SA-4) holds. If* $\frac{nh^d}{\log(1/h)}\to\infty$ *and* $h=o(1)$*, then*

$$\big|\widehat{\vartheta}_{\texttt{WBATE}} - \vartheta_{\texttt{WBATE}}\big| \lesssim_{\mathbb{P}} \big|\mathfrak{B}_{\texttt{WBATE}}\big| + \frac{1}{\sqrt{nh}} + \frac{1}{nh^d}.$$

For inference, define

$$\widehat{\mathrm{T}}_{\mathtt{WBATE}} = \frac{\widehat{\vartheta}_{\mathtt{WBATE}} - \vartheta_{\mathtt{WBATE}}}{\sqrt{\widehat{\Xi}_{\mathtt{WBATE}}}},$$

and the associated confidence interval

$$\widehat{\mathrm{I}}_{\alpha,\mathtt{WBATE}} = \left[\ \widehat{\vartheta}_{\mathtt{WBATE}} - \mathfrak{q}_\alpha\sqrt{\widehat{\Xi}_{\mathtt{WBATE}}}\ ,\ \widehat{\vartheta}_{\mathtt{WBATE}} + \mathfrak{q}_\alpha\sqrt{\widehat{\Xi}_{\mathtt{WBATE}}}\ \right].$$

Theorem SA-12 (Asymptotic Normality: WBATE)**.** *Suppose Assumptions SA-1, SA-2, and SA-3 hold, and (SA-5) and (SA-4) hold. If* $\frac{n^{v/(2+v)}h^d}{\log(1/h)} \to \infty$*,* $h = o(1)$*,* $nh^{2d-1} \to \infty$*, and* $\sqrt{nh}|\mathfrak{B}_{\mathtt{WBATE}}| \to 0$*, then*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\big(\widehat{\mathrm{T}}_{\mathtt{WBATE}} \le u\big) - \Phi(u)\right| = o(1),$$

and

$$\mathbb{P}\big[\vartheta_{\mathtt{WBATE}} \in \widehat{\mathrm{I}}_{\alpha,\mathtt{WBATE}}\big] = 1 - \alpha + o(1)$$

provided that $\mathfrak{q}_\alpha = \Phi^{-1}(1-\alpha/2)$*.*

Condition (SA-4) controls the amount of boundary measure in bandwidth-sized neighborhoods, which gives the effective one-dimensional variance rate $1/(nh)$ for the integrated estimator.

SA-5 Largest Boundary Average Treatment Effect

The distance-based LBATE parameter and plug-in estimator are

$$\vartheta_{\mathtt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \vartheta(\mathbf{x}), \qquad \widehat{\vartheta}_{\mathtt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \widehat{\vartheta}(\mathbf{x}),$$

respectively.

Theorem SA-13 (Convergence Rate: LBATE)**.** *Suppose Assumptions SA-1 and SA-2 hold, and* $\vartheta(\mathbf{x})$ *is well defined for all* $\mathbf{x} \in \mathscr{B}$*. If* $\frac{nh^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$*, then*

$$\left|\widehat{\vartheta}_{\mathtt{LBATE}} - \vartheta_{\mathtt{LBATE}}\right| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d} + \sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}(\mathbf{x})|.$$

For uncertainty quantification, project the simultaneous confidence band for the BATEC onto the supremum functional. Define

$$\widehat{\mathrm{I}}_{\alpha,\mathtt{LBATE}} = \left[\sup_{\mathbf{x}\in\mathscr{B}}\left(\widehat{\vartheta}(\mathbf{x}) - \mathfrak{q}_\alpha\sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}}\right),\quad \sup_{\mathbf{x}\in\mathscr{B}}\left(\widehat{\vartheta}(\mathbf{x}) + \mathfrak{q}_\alpha\sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}}\right)\right],$$

where $\mathfrak{q}_\alpha = \inf\{c > 0 : \mathbb{P}(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{Z}(\mathbf{x})| \ge c \mid \mathbf{V}) \le \alpha\}$.

Theorem SA-14 (Confidence Intervals: LBATE)**.** *Suppose the assumptions and conditions in Theorem SA-*

10 hold. Then

$$\mathbb{P}\left[\vartheta_{\texttt{LBATE}} \in \widehat{\mathrm{I}}_{\alpha,\texttt{LBATE}}\right] \geq 1-\alpha+o(1).$$

The interval is generally conservative because it is obtained by projecting a simultaneous confidence band for the entire distance-based BATEC.

SA-6 Imperfect Compliance

We extend the distance-based results to settings with imperfect compliance, where treatment assignment and treatment receipt may differ for some units. See Hernán and Robins [2020] for a textbook introduction.

Because $D_i(\mathbf{x}) \in \mathscr{I}_t$ if and only if $\mathbf{X}_i \in \mathscr{A}_t$, for any $\mathbf{x} \in \mathscr{B}$ let

$$W_i = \mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_0)W_i(0) + \mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_1)W_i(1)$$

be the observed treatment receipt, where $W_i(t) \in \{0,1\}$ denotes the potential treatment receipt under assignment $t \in \{0,1\}$. The observed outcome is

$$Y_i = \mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_0)Y_i(0, W_i(0)) + \mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_1)Y_i(1, W_i(1)).$$

For notational economy, redefine the reduced-form potential outcome as $Y_i(t) = Y_i(t, W_i(t))$. For $A \in \{Y, W\}$, write $A_i(t) = Y_i(t)$ when $A = Y$, and $A_i(t) = W_i(t)$ when $A = W$; similarly, write $A_i = Y_i$ when $A = Y$, and $A_i = W_i$ when $A = W$. The reduced-form BATEC components are

$$\tau_A(\mathbf{x}) = \mathbb{E}[A_i(1) - A_i(0) \mid \mathbf{X}_i = \mathbf{x}], \qquad A \in \{Y, W\}, \quad \mathbf{x} \in \mathscr{B},$$

and the fuzzy BATEC is the Wald ratio

$$\zeta(\mathbf{x}) = \frac{\tau_Y(\mathbf{x})}{\tau_W(\mathbf{x})}, \qquad \mathbf{x} \in \mathscr{B}.$$

The results below concern this reduced-form Wald target. A complier-average causal interpretation can be obtained by adding the usual exclusion and monotonicity restrictions, which are not needed for the estimation and inference theory.

The distance-based analogues are defined from the induced conditional expectations

$$\theta_{A,t,\mathbf{x}}(r) = \mathbb{E}[A_i \mid D_i(\mathbf{x}) = r] = \mathbb{E}[A_i(t) \mid \mathscr{d}(\mathbf{X}_i, \mathbf{x}) = |r|, \mathbf{X}_i \in \mathscr{A}_t], \qquad r \in \mathscr{I}_t,$$

and

$$\vartheta_A(\mathbf{x}) = \theta_{A,1,\mathbf{x}}(0) - \theta_{A,0,\mathbf{x}}(0), \qquad \xi(\mathbf{x}) = \frac{\vartheta_Y(\mathbf{x})}{\vartheta_W(\mathbf{x})}.$$

The estimator is

$$\widehat{\xi}(\mathbf{x}) = \frac{\widehat{\vartheta}_Y(\mathbf{x})}{\widehat{\vartheta}_W(\mathbf{x})}, \qquad \widehat{\vartheta}_A(\mathbf{x}) = \mathbf{e}_0^\top \widehat{\boldsymbol{\gamma}}_{A,1}(\mathbf{x}) - \mathbf{e}_0^\top \widehat{\boldsymbol{\gamma}}_{A,0}(\mathbf{x}),$$

where, for $A \in \{Y, W\}$ and $t \in \{0, 1\}$,

$$\widehat{\boldsymbol{\gamma}}_{A,t}(\mathbf{x}) = \underset{\boldsymbol{\gamma} \in \mathbb{R}^{p+1}}{\arg\min}\, \mathbb{E}_n\Big[\big(A_i - \mathbf{r}_p(D_i(\mathbf{x})/h)^\top \boldsymbol{\gamma}\big)^2 K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t)\Big].$$

Thus, $\widehat{\boldsymbol{\gamma}}_{Y,t}(\mathbf{x})$ coincides with the sharp-design estimator defined above when the outcome is Y_i, while $\widehat{\boldsymbol{\gamma}}_{W,t}(\mathbf{x})$ is the same local polynomial fit applied to treatment receipt.

Assumption SA-4 (Fuzzy DGP)**.** *Let $t \in \{0, 1\}$.*

(i) $((Y_i(t), W_i(t), \mathbf{X}_i^\top)^\top : 1 \le i \le n)$ *are independent and identically distributed random vectors.*

(ii) $\mu_{W,t}(\mathbf{x}) = \mathbb{E}[W_i(t) \mid \mathbf{X}_i = \mathbf{x}]$ *has a $(p+1)$-times continuously differentiable extension to an open neighborhood of $\mathscr{X}$.*

(iii) *The entries of $\mathbb{C}\mathrm{ov}[(Y_i(t), W_i(t))^\top \mid \mathbf{X}_i = \mathbf{x}]$ are continuous on $\mathscr{X}$.*

The following theorem is a natural consequence of Theorem SA-1.

Theorem SA-15 (Distance-Based Fuzzy Identification)**.** *Suppose Assumptions SA-1(i)–(iii), SA-4, and SA-2(i), (ii), and (v) hold. Then,*

$$\vartheta_A(\mathbf{x}) = \tau_A(\mathbf{x}), \qquad A \in \{Y, W\}, \quad \mathbf{x} \in \mathscr{B}.$$

If, in addition, $\tau_W(\mathbf{x}) \neq 0$, then $\xi(\mathbf{x}) = \zeta(\mathbf{x})$.

SA-6.1 Second-Order Linearization

The distance-based fuzzy analysis starts from the reduced-form distance-based decomposition for each $A \in \{Y, W\}$. Let

$$\theta^*_{A,t,\mathbf{x}}(r) = \mathbf{r}_p(r/h)^\top \boldsymbol{\gamma}^*_{A,t}(\mathbf{x}),$$

where

$$\boldsymbol{\gamma}^*_{A,t}(\mathbf{x}) = \underset{\boldsymbol{\gamma} \in \mathbb{R}^{p+1}}{\arg\min}\, \mathbb{E}\Big[\big(A_i - \mathbf{r}_p(D_i(\mathbf{x})/h)^\top \boldsymbol{\gamma}\big)^2 K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t)\Big].$$

The same algebra used in (SA-1) gives

$$\widehat{\vartheta}_A(\mathbf{x}) - \vartheta_A(\mathbf{x}) = \mathfrak{B}_A(\mathbf{x}) + \mathfrak{L}_A(\mathbf{x}) + \mathfrak{Q}_A(\mathbf{x}), \qquad A \in \{Y, W\}, \tag{SA-6}$$

where

$$\begin{aligned}
\mathfrak{B}_A(\mathbf{x}) &= \theta^*_{A,1,\mathbf{x}}(0) - \theta^*_{A,0,\mathbf{x}}(0) - \vartheta_A(\mathbf{x}),\\
\mathfrak{L}_A(\mathbf{x}) &= \mathbf{e}_0^\top \boldsymbol{\Psi}_{1,\mathbf{x}}^{-1}\mathbf{O}_{A,1,\mathbf{x}} - \mathbf{e}_0^\top \boldsymbol{\Psi}_{0,\mathbf{x}}^{-1}\mathbf{O}_{A,0,\mathbf{x}},\\
\mathfrak{Q}_A(\mathbf{x}) &= \mathbf{e}_0^\top (\widehat{\boldsymbol{\Psi}}_{1,\mathbf{x}}^{-1} - \boldsymbol{\Psi}_{1,\mathbf{x}}^{-1})\mathbf{O}_{A,1,\mathbf{x}} - \mathbf{e}_0^\top (\widehat{\boldsymbol{\Psi}}_{0,\mathbf{x}}^{-1} - \boldsymbol{\Psi}_{0,\mathbf{x}}^{-1})\mathbf{O}_{A,0,\mathbf{x}},
\end{aligned}$$

and

$$\mathbf{O}_{A,t,\mathbf{x}} = \mathbb{E}_n\Big[\mathbf{r}_p(D_i(\mathbf{x})/h)K_h(D_i(\mathbf{x}))\big(A_i - \theta^*_{A,t,\mathbf{x}}(D_i(\mathbf{x}))\big)\mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t)\Big].$$

The covariance kernel for the reduced-form linear terms is defined as follows. For $A, B \in \{Y, W\}$, let

$$\Xi_{A,B,\mathbf{x}_1,\mathbf{x}_2} = \Xi_{A,B,0,\mathbf{x}_1,\mathbf{x}_2} + \Xi_{A,B,1,\mathbf{x}_1,\mathbf{x}_2},$$

where

$$\Xi_{A,B,t,\mathbf{x}_1,\mathbf{x}_2} = \frac{1}{nh^d}\mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}_1}^{-1} \boldsymbol{\Upsilon}_{A,B,t,\mathbf{x}_1,\mathbf{x}_2} \boldsymbol{\Psi}_{t,\mathbf{x}_2}^{-1} \mathbf{e}_0,$$

and

$$\begin{aligned}\boldsymbol{\Upsilon}_{A,B,t,\mathbf{x}_1,\mathbf{x}_2} = h^d \mathbb{E}\Big[&\mathbf{r}_p(D_i(\mathbf{x}_1)/h)\mathbf{r}_p(D_i(\mathbf{x}_2)/h)^\top K_h(D_i(\mathbf{x}_1))K_h(D_i(\mathbf{x}_2)) \\ &\times \big(A_i - \theta^*_{A,t,\mathbf{x}_1}(D_i(\mathbf{x}_1))\big)\big(B_i - \theta^*_{B,t,\mathbf{x}_2}(D_i(\mathbf{x}_2))\big)\mathbb{1}(D_i(\mathbf{x}_1) \in \mathscr{I}_t)\mathbb{1}(D_i(\mathbf{x}_2) \in \mathscr{I}_t)\Big].\end{aligned}$$

Let $\widehat{\Xi}_{A,B,\mathbf{x}_1,\mathbf{x}_2}$ denote the sample analogue obtained by replacing $\boldsymbol{\Psi}$, $\boldsymbol{\Upsilon}$, and θ^* with $\widehat{\boldsymbol{\Psi}}$, $\widehat{\boldsymbol{\Upsilon}}$, and $\widehat{\theta}$.

The exact second-order Wald expansion is

$$\widehat{\xi}(\mathbf{x}) - \xi(\mathbf{x}) = \mathfrak{B}_{\mathrm{F}}(\mathbf{x}) + \mathfrak{L}_{\mathrm{F}}(\mathbf{x}) + \mathfrak{Q}_{\mathrm{F}}(\mathbf{x}) + \mathfrak{R}_{\mathrm{F}}(\mathbf{x}), \tag{SA-7}$$

where

$$\begin{aligned}\mathfrak{B}_{\mathrm{F}}(\mathbf{x}) &= \mathfrak{w}(\mathbf{x})^\top \big(\mathfrak{B}_Y(\mathbf{x}), \mathfrak{B}_W(\mathbf{x})\big)^\top, \\ \mathfrak{L}_{\mathrm{F}}(\mathbf{x}) &= \mathfrak{w}(\mathbf{x})^\top \big(\mathfrak{L}_Y(\mathbf{x}), \mathfrak{L}_W(\mathbf{x})\big)^\top, \\ \mathfrak{Q}_{\mathrm{F}}(\mathbf{x}) &= \mathfrak{w}(\mathbf{x})^\top \big(\mathfrak{Q}_Y(\mathbf{x}), \mathfrak{Q}_W(\mathbf{x})\big)^\top,\end{aligned}$$

and

$$\mathfrak{w}(\mathbf{x}) = \Big(\frac{1}{\vartheta_W(\mathbf{x})}, -\frac{\vartheta_Y(\mathbf{x})}{\vartheta_W(\mathbf{x})^2}\Big)^\top,$$

with second-order remainder

$$\mathfrak{R}_{\mathrm{F}}(\mathbf{x}) = \frac{\vartheta_Y(\mathbf{x})}{\vartheta_W(\mathbf{x})^2\widehat{\vartheta}_W(\mathbf{x})}\big(\widehat{\vartheta}_W(\mathbf{x}) - \vartheta_W(\mathbf{x})\big)^2 - \frac{1}{\vartheta_W(\mathbf{x})\widehat{\vartheta}_W(\mathbf{x})}\big(\widehat{\vartheta}_Y(\mathbf{x}) - \vartheta_Y(\mathbf{x})\big)\big(\widehat{\vartheta}_W(\mathbf{x}) - \vartheta_W(\mathbf{x})\big).$$

If $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$, $h = o(1)$, and $\frac{\log(1/h)}{n^{(1+v)/(2+v)}h^d} = o(1)$, then

$$\sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{R}_{\mathrm{F}}(\mathbf{x})| \lesssim_{\mathbb{P}} \rho_n^2, \tag{SA-8}$$

where

$$\rho_n = \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d} + \sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}_Y(\mathbf{x})| + \sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}_W(\mathbf{x})|.$$

Thus, after the reduced-form distance-based decompositions are in place, the fuzzy BATEC estimator is governed by a fixed linear combination of the reduced-form outcome and first-stage components, up to the higher-order Wald remainder.

SA-6.2 Fuzzy BATEC

SA-6.2.1 Point Estimation and Approximate MSE Expansions

Theorem SA-16 (Convergence Rates: Fuzzy BATEC)**.** *Suppose Assumptions SA-1, SA-4, and SA-2 hold, and let $h = o(1)$.*

- *If $\xi(\mathbf{x})$ is well defined at a fixed $\mathbf{x} \in \mathscr{B}$, $|\vartheta_W(\mathbf{x})| > 0$, and $n^{\frac{1+v}{2+v}} h^d \to \infty$, then*

$$|\widehat{\xi}(\mathbf{x}) - \xi(\mathbf{x})| \lesssim_{\mathbb{P}} \frac{1}{\sqrt{nh^d}} + \frac{1}{n^{\frac{1+v}{2+v}} h^d} + |\mathfrak{B}_Y(\mathbf{x})| + |\mathfrak{B}_W(\mathbf{x})|.$$

- *If $\xi(\mathbf{x})$ is well defined for all $\mathbf{x} \in \mathscr{B}$, $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$, and $\frac{\log(1/h)}{n^{\frac{1+v}{2+v}} h^d} = o(1)$, then*

$$\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{\xi}(\mathbf{x}) - \xi(\mathbf{x})| \lesssim_{\mathbb{P}} \rho_n.$$

Define

$$\mathbf{\Theta}_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} = \begin{pmatrix} \Xi_{Y,Y,\mathbf{x}_1,\mathbf{x}_2} & \Xi_{Y,W,\mathbf{x}_1,\mathbf{x}_2} \\ \Xi_{W,Y,\mathbf{x}_1,\mathbf{x}_2} & \Xi_{W,W,\mathbf{x}_1,\mathbf{x}_2} \end{pmatrix}, \qquad \Xi_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} = \mathfrak{w}(\mathbf{x}_1)^\top \mathbf{\Theta}_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} \mathfrak{w}(\mathbf{x}_2).$$

Then $\mathbb{E}[\mathfrak{L}_{\mathrm{F}}(\mathbf{x})] = 0$ and $\mathbb{E}[\mathfrak{L}_{\mathrm{F}}(\mathbf{x})^2] = \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}$. The linearized approximate MSE and approximate IMSE are

$$\mathrm{AMSE}_{\mathrm{F}}(\mathbf{x}) = \mathfrak{B}_{\mathrm{F}}(\mathbf{x})^2 + \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \qquad \text{and} \qquad \mathrm{AIMSE}_{\mathrm{F}} = \int_{\mathscr{B}} \mathrm{AMSE}_{\mathrm{F}}(\mathbf{x}) w(\mathbf{x}) \, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).$$

These criteria omit the higher-order terms $\mathfrak{Q}_{\mathrm{F}}(\mathbf{x})$ and $\mathfrak{R}_{\mathrm{F}}(\mathbf{x})$. If a leading-bias expansion $\mathfrak{B}_{\mathrm{F}}(\mathbf{x}) = h^m B_{\mathrm{F},\mathbf{x}} + o(h^m)$ is available and $V_{\mathrm{F},\mathbf{x}} = nh^d \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}$, then the approximate pointwise criterion is

$$h^{2m} B_{\mathrm{F},\mathbf{x}}^2 + \frac{1}{nh^d} V_{\mathrm{F},\mathbf{x}},$$

with the same bandwidth formulas as in the sharp distance-based case after replacing $(B_{\mathbf{x}}, V_{\mathbf{x}})$ by $(B_{\mathrm{F},\mathbf{x}}, V_{\mathrm{F},\mathbf{x}})$.

SA-6.2.2 Distributional Approximation and Inference

Let

$$\widehat{\mathfrak{w}}(\mathbf{x}) = \Big(\frac{1}{\widehat{\vartheta}_W(\mathbf{x})}, -\frac{\widehat{\vartheta}_Y(\mathbf{x})}{\widehat{\vartheta}_W(\mathbf{x})^2} \Big)^\top,$$

and define

$$\widehat{\mathbf{\Theta}}_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} = \begin{pmatrix} \widehat{\Xi}_{Y,Y,\mathbf{x}_1,\mathbf{x}_2} & \widehat{\Xi}_{Y,W,\mathbf{x}_1,\mathbf{x}_2} \\ \widehat{\Xi}_{W,Y,\mathbf{x}_1,\mathbf{x}_2} & \widehat{\Xi}_{W,W,\mathbf{x}_1,\mathbf{x}_2} \end{pmatrix}, \qquad \widehat{\Xi}_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} = \widehat{\mathfrak{w}}(\mathbf{x}_1)^\top \widehat{\mathbf{\Theta}}_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} \widehat{\mathfrak{w}}(\mathbf{x}_2).$$

Lemma SA-8 (Covariance: Fuzzy BATEC)**.** *Suppose Assumptions SA-1, SA-4, and SA-2 hold,* $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$*, and* $1 \lesssim \inf_{\mathbf{x}\in\mathscr{B}} nh^d \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \leq \sup_{\mathbf{x}\in\mathscr{B}} nh^d \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \lesssim 1$*. If* $\frac{\log(1/h)}{n^{v/(2+v)} h^d} = o(1)$ *and* $h = o(1)$*, then*

$$\sup_{\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}} nh^d \big| \widehat{\Xi}_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} - \Xi_{\mathrm{F},\mathbf{x}_1,\mathbf{x}_2} \big| = o_{\mathbb{P}}(1) \qquad \textit{and} \qquad \sup_{\mathbf{x}\in\mathscr{B}} \left| \frac{\widehat{\Xi}_{\mathrm{F},\mathbf{x},\mathbf{x}}}{\Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}} - 1 \right| = o_{\mathbb{P}}(1).$$

The feasible t-statistic and confidence interval are

$$\widehat{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}) = \frac{\widehat{\xi}(\mathbf{x}) - \xi(\mathbf{x})}{\sqrt{\widehat{\Xi}_{\mathrm{F},\mathbf{x},\mathbf{x}}}} \qquad \text{and} \qquad \widehat{\mathrm{I}}_{\alpha,\mathrm{F}}(\mathbf{x}) = \left[\widehat{\xi}(\mathbf{x}) - \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathrm{F},\mathbf{x},\mathbf{x}}}, \widehat{\xi}(\mathbf{x}) + \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathrm{F},\mathbf{x},\mathbf{x}}}\right].$$

For the pointwise remainder condition, define

$$\rho_{n,\mathbf{x}} = \frac{1}{\sqrt{nh^d}} + \frac{1}{n^{\frac{1+v}{2+v}} h^d} + |\mathfrak{B}_Y(\mathbf{x})| + |\mathfrak{B}_W(\mathbf{x})|.$$

Theorem SA-17 (Confidence Intervals: Fuzzy BATEC)**.** *For a fixed boundary point $\mathbf{x} \in \mathscr{B}$, suppose Assumptions SA-1, SA-4, and SA-2 hold, $|\vartheta_W(\mathbf{x})| > 0$, and $1 \lesssim nh^d \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \lesssim 1$. If $n^{v/(2+v)} h^d \to \infty$, $h = o(1)$, $\sqrt{nh^d}\,|\mathfrak{B}_{\mathrm{F}}(\mathbf{x})| \to 0$, and $\sqrt{nh^d}\rho_{n,\mathbf{x}}^2 \to 0$, then*

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}\big(\widehat{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}) \le u\big) - \Phi(u) \right| = o(1),$$

and

$$\mathbb{P}\big(\xi(\mathbf{x}) \in \widehat{\mathrm{I}}_{\alpha,\mathrm{F}}(\mathbf{x})\big) = 1 - \alpha + o(1),$$

provided that $\mathfrak{q}_\alpha = \Phi^{-1}(1-\alpha/2)$.

For uniform inference, define the infeasible linearized statistic

$$\overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}) = \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}^{-1/2} \mathfrak{L}_{\mathrm{F}}(\mathbf{x}), \qquad \mathbf{x} \in \mathscr{B}.$$

Theorem SA-18 (Stochastic Linearization: Fuzzy BATEC)**.** *Suppose the assumptions of Lemma SA-8 hold. Then*

$$\begin{aligned}\sup_{\mathbf{x}\in\mathscr{B}} \left|\widehat{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}) - \overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x})\right| \lesssim_{\mathbb{P}} \sqrt{\log(1/h)}\left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{v/(2+v)}h^d}\right) \\ + \sqrt{nh^d} \sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}_{\mathrm{F}}(\mathbf{x})| + \sqrt{nh^d}\rho_n^2.\end{aligned}$$

Theorem SA-19 (Gaussian Approximation for $\sup_{\mathbf{x}\in\mathscr{B}} |\overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x})|$)**.** *Suppose Assumptions SA-1, SA-4, and SA-2 hold, $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$, $1 \lesssim \inf_{\mathbf{x}\in\mathscr{B}} nh^d \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}$, $\sup_{\mathbf{x}\in\mathscr{B}} nh^d \Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \lesssim 1$, $\liminf_{n\to\infty} \frac{\log h}{\log n} > -\infty$, and $h = o(1)$. Then, for n large enough, on a possibly enlarged probability space there exists a mean-zero Gaussian process Z_{F} indexed by $\mathscr{B}$ with almost surely continuous sample paths such that*

$$\mathbb{C}\mathrm{ov}[Z_{\mathrm{F}}(\mathbf{x}_1), Z_{\mathrm{F}}(\mathbf{x}_2)] = \mathbb{C}\mathrm{ov}[\overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}_1), \overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}_2)], \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathscr{B},$$

and

$$\mathbb{P}\left[\left|\sup_{\mathbf{x}\in\mathscr{B}} |\overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x})| - \sup_{\mathbf{x}\in\mathscr{B}} |Z_{\mathrm{F}}(\mathbf{x})|\right| > \frac{\log n}{(nh^d)^{1/6}} + \frac{(\log n)^{5/4}}{(nh^d)^{1/4}} + \frac{(\log n)^{3/2}}{(n^{v/(2+v)}h^d)^{1/2}}\right] \lesssim \frac{1}{\log n} + \frac{\log n}{n}.$$

Let $(\widehat{Z}_{\mathrm{F}}(\mathbf{x}) : \mathbf{x} \in \mathscr{B})$ be a mean-zero Gaussian process conditional on $\mathbf{V} = ((\mathbf{X}_i^\top, Y_i, W_i) : 1 \le i \le n)$ with

covariance

$$\mathbb{C}\mathrm{ov}[\widehat{Z}_{\mathtt{F}}(\mathbf{x}_1),\widehat{Z}_{\mathtt{F}}(\mathbf{x}_2)\mid \mathbf{V}]=\frac{\widehat{\Xi}_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}}{\sqrt{\widehat{\Xi}_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_1}}\sqrt{\widehat{\Xi}_{\mathtt{F},\mathbf{x}_2,\mathbf{x}_2}}},\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}.$$

Theorem SA-20 (Confidence Bands: Fuzzy BATEC)**.** *Suppose the assumptions of Lemma SA-8 hold. If* $\liminf_{n\to\infty}\frac{\log h}{\log n}>-\infty$, $\frac{n^{v/(2+v)}h^d}{(\log n)^9}\to\infty$, $h=o(1)$, $\sqrt{nh^d\log n}\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}_{\mathtt{F}}(\mathbf{x})|\to 0$, *and* $\sqrt{nh^d\log n}\rho_n^2\to 0$, *then*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}_{\mathtt{F}}(\mathbf{x})|\le u\right)-\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{Z}_{\mathtt{F}}(\mathbf{x})|\le u\mid\mathbf{V}\right)\right|=o_{\mathbb{P}}(1),$$

and

$$\mathbb{P}\big[\xi(\mathbf{x})\in\widehat{\mathrm{I}}_{\alpha,\mathtt{F}}(\mathbf{x}),\ \textit{for all}\ \mathbf{x}\in\mathscr{B}\big]=1-\alpha+o(1),$$

provided that $\mathfrak{q}_\alpha=\inf\{c>0:\mathbb{P}(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{Z}_{\mathtt{F}}(\mathbf{x})|\ge c\mid\mathbf{V})\le\alpha\}$.

SA-6.3 Fuzzy WBATE

We impose the normalization $\int_{\mathscr{B}}w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})=1$. The distance-based fuzzy WBATE parameter and plug-in estimator are

$$\xi_{\mathtt{WBATE}}=\int_{\mathscr{B}}\xi(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})\qquad\text{and}\qquad\widehat{\xi}_{\mathtt{WBATE}}=\int_{\mathscr{B}}\widehat{\xi}(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).$$

Under the conditions of Theorem SA-15, $\xi_{\mathtt{WBATE}}=\zeta_{\mathtt{WBATE}}$, where $\zeta_{\mathtt{WBATE}}=\int_{\mathscr{B}}\zeta(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})$.

Integrating the pointwise fuzzy expansion (SA-7) gives

$$\widehat{\xi}_{\mathtt{WBATE}}-\xi_{\mathtt{WBATE}}=\mathfrak{B}_{\mathtt{F},\mathtt{WBATE}}+\mathfrak{L}_{\mathtt{F},\mathtt{WBATE}}+\mathfrak{Q}_{\mathtt{F},\mathtt{WBATE}}+\mathfrak{R}_{\mathtt{F},\mathtt{WBATE}},\tag{SA-9}$$

where

$$\begin{aligned}\mathfrak{B}_{\mathtt{F},\mathtt{WBATE}}&=\int_{\mathscr{B}}\mathfrak{B}_{\mathtt{F}}(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}),\\ \mathfrak{L}_{\mathtt{F},\mathtt{WBATE}}&=\int_{\mathscr{B}}\mathfrak{L}_{\mathtt{F}}(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}),\\ \mathfrak{Q}_{\mathtt{F},\mathtt{WBATE}}&=\int_{\mathscr{B}}\mathfrak{Q}_{\mathtt{F}}(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}),\\ \mathfrak{R}_{\mathtt{F},\mathtt{WBATE}}&=\int_{\mathscr{B}}\mathfrak{R}_{\mathtt{F}}(\mathbf{x})w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).\end{aligned}$$

The aggregate variance of the leading linearized term is

$$\Xi_{\mathtt{F},\mathtt{WBATE}}=\int_{\mathscr{B}}\int_{\mathscr{B}}\Xi_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}w(\mathbf{x}_1)w(\mathbf{x}_2)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_1)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_2),$$

with feasible counterpart

$$\widehat{\Xi}_{\mathtt{F},\mathtt{WBATE}}=\int_{\mathscr{B}}\int_{\mathscr{B}}\widehat{\Xi}_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}w(\mathbf{x}_1)w(\mathbf{x}_2)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_1)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_2).$$

For inference, impose the aggregate nondegeneracy condition

$$\Xi_{\mathtt{F,WBATE}} \gtrsim \frac{1}{nh}, \tag{SA-10}$$

which rules out cancellation in the integrated fuzzy covariance kernel.

Lemma SA-9 (Variance Linear Term: Fuzzy WBATE)**.** *Suppose Assumptions SA-1, SA-4, SA-2, and SA-3 hold,* $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$*, and* (SA-4) *holds. If* $\frac{nh^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$*, then*

$$\mathbb{V}[\mathfrak{L}_{\mathtt{F,WBATE}}] = \Xi_{\mathtt{F,WBATE}} \lesssim \frac{1}{nh}.$$

If, in addition, (SA-10) *holds, then*

$$\frac{1}{nh} \lesssim \Xi_{\mathtt{F,WBATE}} \lesssim \frac{1}{nh}.$$

If $\frac{n^{v/(2+v)}h^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$*, then*

$$\widehat{\Xi}_{\mathtt{F,WBATE}} = \Xi_{\mathtt{F,WBATE}} + o_{\mathbb{P}}\Big(\frac{1}{nh}\Big).$$

If, in addition, (SA-10) *holds, then* $\widehat{\Xi}_{\mathtt{F,WBATE}}/\Xi_{\mathtt{F,WBATE}} \to_{\mathbb{P}} 1$*.*

Lemma SA-10 (Higher-Order Remainders: Fuzzy WBATE)**.** *Suppose Assumptions SA-1, SA-4, SA-2, and SA-3 hold,* $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$*, and* (SA-4) *holds. If* $\frac{nh^d}{\log(1/h)} \to \infty$ *and* $h = o(1)$*, then*

$$\mathfrak{Q}_{\mathtt{F,WBATE}} \lesssim_{\mathbb{P}} \frac{1}{nh^d}.$$

If, in addition, $\frac{\log(1/h)}{n^{(1+v)/(2+v)}h^d} = o(1)$*, then*

$$|\mathfrak{R}_{\mathtt{F,WBATE}}| \lesssim_{\mathbb{P}} \rho_n^2.$$

Consequently, if also $nh^{2d-1} \to \infty$ *and* $\sqrt{nh}\rho_n^2 \to 0$*, then*

$$\mathfrak{Q}_{\mathtt{F,WBATE}} + \mathfrak{R}_{\mathtt{F,WBATE}} = o_{\mathbb{P}}\Big(\frac{1}{\sqrt{nh}}\Big).$$

Theorem SA-21 (Convergence Rate: Fuzzy WBATE)**.** *Suppose Assumptions SA-1, SA-4, SA-2, and SA-3 hold,* $\xi(\mathbf{x})$ *is well defined for all* $\mathbf{x} \in \mathscr{B}$*,* $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$*, and* (SA-4) *holds. If* $\frac{\log(1/h)}{n^{(1+v)/(2+v)}h^d} = o(1)$ *and* $h = o(1)$*, then*

$$\Big|\widehat{\xi}_{\mathtt{WBATE}} - \xi_{\mathtt{WBATE}}\Big| \lesssim_{\mathbb{P}} |\mathfrak{B}_{\mathtt{F,WBATE}}| + \frac{1}{\sqrt{nh}} + \frac{1}{nh^d} + \rho_n^2.$$

For inference, define

$$\widehat{\mathrm{T}}_{\mathtt{F,WBATE}} = \frac{\widehat{\xi}_{\mathtt{WBATE}} - \xi_{\mathtt{WBATE}}}{\sqrt{\widehat{\Xi}_{\mathtt{F,WBATE}}}},$$

and

$$\widehat{\mathrm{I}}_{\alpha,\mathtt{F},\mathtt{WBATE}} = \left[\widehat{\xi}_{\mathtt{WBATE}} - \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathtt{F},\mathtt{WBATE}}}, \quad \widehat{\xi}_{\mathtt{WBATE}} + \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathtt{F},\mathtt{WBATE}}}\right].$$

Theorem SA-22 (Asymptotic Normality: Fuzzy WBATE)**.** *Suppose Assumptions SA-1, SA-4, SA-2, and SA-3 hold, $\xi(\mathbf{x})$ is well defined for all $\mathbf{x} \in \mathscr{B}$, $\inf_{\mathbf{x}\in\mathscr{B}} |\vartheta_W(\mathbf{x})| > 0$, and (SA-10) and (SA-4) hold. If $\frac{\log(1/h)}{n^{v/(2+v)}h^d} = o(1)$, $h = o(1)$, $nh^{2d-1} \to \infty$, $\sqrt{nh}|\mathfrak{B}_{\mathtt{F},\mathtt{WBATE}}| \to 0$, and $\sqrt{nh}\rho_n^2 \to 0$, then*

$$\sup_{u\in\mathbb{R}} \left| \mathbb{P}\big(\widehat{\mathrm{T}}_{\mathtt{F},\mathtt{WBATE}} \le u\big) - \Phi(u) \right| = o(1),$$

and

$$\mathbb{P}\big[\xi_{\mathtt{WBATE}} \in \widehat{\mathrm{I}}_{\alpha,\mathtt{F},\mathtt{WBATE}}\big] = 1 - \alpha + o(1),$$

provided that $\mathfrak{q}_\alpha = \Phi^{-1}(1-\alpha/2)$.

SA-6.4 Fuzzy LBATE

The distance-based fuzzy LBATE parameter and plug-in estimator are

$$\xi_{\mathtt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \xi(\mathbf{x}) \qquad \text{and} \qquad \widehat{\xi}_{\mathtt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \widehat{\xi}(\mathbf{x}).$$

Under the conditions of Theorem SA-15, $\xi_{\mathtt{LBATE}} = \zeta_{\mathtt{LBATE}}$, where $\zeta_{\mathtt{LBATE}} = \sup_{\mathbf{x}\in\mathscr{B}} \zeta(\mathbf{x})$.

Theorem SA-23 (Convergence Rate: Fuzzy LBATE)**.** *Suppose the assumptions and conditions of the uniform part of Theorem SA-16 hold. Then*

$$\left|\widehat{\xi}_{\mathtt{LBATE}} - \xi_{\mathtt{LBATE}}\right| \lesssim_{\mathbb{P}} \rho_n.$$

Projecting the fuzzy BATEC confidence band gives the fuzzy LBATE confidence interval

$$\widehat{\mathrm{I}}_{\alpha,\mathtt{F},\mathtt{LBATE}} = \left[\sup_{\mathbf{x}\in\mathscr{B}} \left(\widehat{\xi}(\mathbf{x}) - \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathtt{F},\mathbf{x},\mathbf{x}}}\right), \quad \sup_{\mathbf{x}\in\mathscr{B}} \left(\widehat{\xi}(\mathbf{x}) + \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\mathtt{F},\mathbf{x},\mathbf{x}}}\right)\right],$$

where $\mathfrak{q}_\alpha = \inf\{c > 0 : \mathbb{P}(\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{Z}_{\mathtt{F}}(\mathbf{x})| \ge c \mid \mathbf{V}) \le \alpha\}$.

Theorem SA-24 (Confidence Intervals: Fuzzy LBATE)**.** *Suppose the assumptions and conditions in Theorem SA-20 hold. Then*

$$\mathbb{P}\left[\xi_{\mathtt{LBATE}} \in \widehat{\mathrm{I}}_{\alpha,\mathtt{F},\mathtt{LBATE}}\right] \ge 1 - \alpha + o(1).$$

As in the sharp-design LBATE case, this interval is generally conservative because it is obtained by projecting a simultaneous confidence band for the full fuzzy BATEC.

SA-7 Gaussian Strong Approximation

We present a Gaussian strong approximation theorem, which is the key technical tool behind Theorem SA-7. The theorem builds on and generalizes the results in Cattaneo and Yu [2025]. Consider the *residual-based empirical process* given by

$$M_n[g,r] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big[g(\mathbf{x}_i)r(y_i) - \mathbb{E}[g(\mathbf{x}_i)r(y_i)]\Big], \qquad g \in \mathscr{G}, r \in \mathscr{R},$$

where $\mathscr{G}$ and $\mathscr{R}$ are classes of functions satisfying certain regularity conditions.

SA-7.1 Definitions for Function Spaces

Let $\mathscr{F}$ be a class of measurable functions from a probability space $(\mathbb{R}^q, \mathscr{B}(\mathbb{R}^q), \mathbb{P})$ to $\mathbb{R}$. We introduce several definitions that capture properties of $\mathscr{F}$.

(i) $\mathscr{F}$ is pointwise measurable if it contains a countable subset $\mathscr{G}$ such that for any $f \in \mathscr{F}$, there exists a sequence $(g_m : m \geq 1) \subseteq \mathscr{G}$ such that $\lim_{m\to\infty} g_m(\mathbf{u}) = f(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^q$.

(ii) Let $\operatorname{supp}(\mathscr{F}) = \cup_{f\in\mathscr{F}} \operatorname{supp}(f)$. A probability measure $\mathbb{Q}_{\mathscr{F}}$ on $(\mathbb{R}^q, \mathscr{B}(\mathbb{R}^q))$ is a surrogate measure for $\mathbb{P}$ with respect to $\mathscr{F}$ if

(i) $\mathbb{Q}_{\mathscr{F}}$ agrees with $\mathbb{P}$ on $\operatorname{supp}(\mathbb{P}) \cap \operatorname{supp}(\mathscr{F})$.

(ii) $\mathbb{Q}_{\mathscr{F}}(\operatorname{supp}(\mathscr{F}) \setminus \operatorname{supp}(\mathbb{P})) = 0$.

Let $\mathscr{Q}_{\mathscr{F}} = \operatorname{supp}(\mathbb{Q}_{\mathscr{F}})$.

(iii) For $q = 1$ and an interval $\mathscr{I} \subseteq \mathbb{R}$, the pointwise total variation of $\mathscr{F}$ over $\mathscr{I}$ is

$$\mathtt{pTV}_{\mathscr{F},\mathscr{I}} = \sup_{f\in\mathscr{F}}\sup_{P\geq 1}\sup_{\mathscr{P}_P\in\mathscr{I}}\sum_{i=1}^{P-1}|f(a_{i+1}) - f(a_i)|,$$

where $\mathscr{P}_P = \{(a_1,\ldots,a_P) : a_1 \leq \cdots \leq a_P\}$ denotes the collection of all partitions of $\mathscr{I}$.

(iv) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the total variation of $\mathscr{F}$ over $\mathscr{C}$ is

$$\mathtt{TV}_{\mathscr{F},\mathscr{C}} = \inf_{\mathscr{U}\in\mathscr{O}(\mathscr{C})}\sup_{f\in\mathscr{F}}\sup_{\phi\in\mathscr{D}_q(\mathscr{U})}\int_{\mathbb{R}^q} f(\mathbf{u})\operatorname{div}(\phi)(\mathbf{u})\,\mathrm{d}\mathbf{u}/\|\|\phi\|_2\|_\infty,$$

where $\mathscr{O}(\mathscr{C})$ denotes the collection of all open sets that contains $\mathscr{C}$, and $\mathscr{D}_q(\mathscr{U})$ denotes the space of infinitely differentiable functions from $\mathbb{R}^q$ to $\mathbb{R}^q$ with compact support contained in $\mathscr{U}$.

(v) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the local total variation constant of $\mathscr{F}$ over $\mathscr{C}$, is a positive number $\mathtt{K}_{\mathscr{F},\mathscr{C}}$ such that for any cube $\mathscr{D} \subseteq \mathbb{R}^q$ with edges of length ℓ parallel to the coordinate axises,

$$\mathtt{TV}_{\mathscr{F},\mathscr{D}\cap\mathscr{C}} \leq \mathtt{K}_{\mathscr{F},\mathscr{C}}\ell^{d-1}.$$

(vi) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the envelopes of $\mathscr{F}$ over $\mathscr{C}$ are

$$\mathtt{M}_{\mathscr{F},\mathscr{C}} = \sup_{\mathbf{u}\in\mathscr{C}} M_{\mathscr{F},\mathscr{C}}(\mathbf{u}), \qquad M_{\mathscr{F},\mathscr{C}}(\mathbf{u}) = \sup_{f\in\mathscr{F}}|f(\mathbf{u})|, \qquad \mathbf{u} \in \mathscr{C}.$$

(vii) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the Lipschitz constant of $\mathscr{F}$ over $\mathscr{C}$ is

$$\mathtt{L}_{\mathscr{F},\mathscr{C}} = \sup_{f\in\mathscr{F}} \sup_{\mathbf{u}_1,\mathbf{u}_2\in\mathscr{C}} \frac{|f(\mathbf{u}_1)-f(\mathbf{u}_2)|}{\|\mathbf{u}_1-\mathbf{u}_2\|_\infty}.$$

(viii) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the L_1 bound of $\mathscr{F}$ over $\mathscr{C}$ is

$$\mathtt{E}_{\mathscr{F},\mathscr{C}} = \sup_{f\in\mathscr{F}} \int_{\mathscr{C}} |f| \,\mathrm{d}\mathbb{P}.$$

(ix) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the uniform covering number of $\mathscr{F}$ with envelope $M_{\mathscr{F},\mathscr{C}}$ over $\mathscr{C}$ is

$$\mathtt{N}_{\mathscr{F},\mathscr{C}}(\delta, M_{\mathscr{F},\mathscr{C}}) = \sup_{\mu} N(\mathscr{F}, \|\cdot\|_{\mu,2}, \delta \|M_{\mathscr{F},\mathscr{C}}\|_{\mu,2}), \qquad \delta \in (0,\infty),$$

where the supremum is taken over all finite discrete measures on $(\mathscr{C}, \mathscr{B}(\mathscr{C}))$. We assume that $M_{\mathscr{F},\mathscr{C}}(\mathbf{u})$ is finite for every $\mathbf{u} \in \mathscr{C}$.

(x) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, the uniform entropy integral of $\mathscr{F}$ with envelope $M_{\mathscr{F},\mathscr{C}}$ over $\mathscr{C}$ is

$$J_{\mathscr{C}}(\delta, \mathscr{F}, M_{\mathscr{F},\mathscr{C}}) = \int_0^\delta \sqrt{1 + \log \mathtt{N}_{\mathscr{F},\mathscr{C}}(\varepsilon, M_{\mathscr{F},\mathscr{C}})} \,\mathrm{d}\varepsilon,$$

where it is assumed that $M_{\mathscr{F},\mathscr{C}}(\mathbf{u})$ is finite for every $\mathbf{u} \in \mathscr{C}$.

(xi) For a non-empty $\mathscr{C} \subseteq \mathbb{R}^q$, $\mathscr{F}$ is a VC-type class with envelope $M_{\mathscr{F},\mathscr{C}}$ over $\mathscr{C}$ if (i) $M_{\mathscr{F},\mathscr{C}}$ is measurable and $M_{\mathscr{F},\mathscr{C}}(\mathbf{u})$ is finite for every $\mathbf{u} \in \mathscr{C}$, and (ii) there exist $\mathtt{c}_{\mathscr{F},\mathscr{C}} > 0$ and $\mathtt{d}_{\mathscr{F},\mathscr{C}} > 0$ such that

$$\mathtt{N}_{\mathscr{F},\mathscr{C}}(\varepsilon, M_{\mathscr{F},\mathscr{C}}) \leq \mathtt{c}_{\mathscr{F},\mathscr{C}} \varepsilon^{-\mathtt{d}_{\mathscr{F},\mathscr{C}}}, \qquad \varepsilon \in (0,1).$$

If a surrogate measure $\mathbb{Q}_{\mathscr{F}}$ for $\mathbb{P}$ with respect to $\mathscr{F}$ has been assumed, and it is clear from the context, we drop the dependence on $\mathscr{C} = \mathscr{Q}_{\mathscr{F}}$ for all quantities in the previous definitions. That is, to save notation, we set $\mathtt{TV}_{\mathscr{F}} = \mathtt{TV}_{\mathscr{F},\mathscr{Q}_{\mathscr{F}}}$, $\mathtt{K}_{\mathscr{F}} = \mathtt{K}_{\mathscr{F},\mathscr{Q}_{\mathscr{F}}}$, $\mathtt{M}_{\mathscr{F}} = \mathtt{M}_{\mathscr{F},\mathscr{Q}_{\mathscr{F}}}$, $M_{\mathscr{F}}(\mathbf{u}) = M_{\mathscr{F},\mathscr{Q}_{\mathscr{F}}}(\mathbf{u})$, $\mathtt{L}_{\mathscr{F}} = \mathtt{L}_{\mathscr{F},\mathscr{Q}_{\mathscr{F}}}$, and so on, whenever there is no confusion.

SA-7.2 Multiplicative-Separable Empirical Process

The following theorem generalizes Cattaneo and Yu [2025, Theorem SA.1] by requiring only bounded polynomial moments for y_i conditional on $\mathbf{x}_i$.

Theorem SA-25 (Strong Approximation for $(M_n(g,r)+M_n(h,s) : g \in \mathscr{G}, r \in \mathscr{R}, h \in \mathscr{H}, s \in \mathscr{S})$)**.** *Suppose $(\mathbf{z}_i = (\mathbf{x}_i, y_i) : 1 \leq i \leq n)$ are i.i.d. random vectors taking values in $(\mathbb{R}^{d+1}, \mathscr{B}(\mathbb{R}^{d+1}))$ with common law $\mathbb{P}_Z$, where $\mathbf{x}_i$ has distribution $\mathbb{P}_X$ supported on $\mathscr{X} \subseteq \mathbb{R}^d$, y_i has distribution $\mathbb{P}_Y$ supported on $\mathscr{Y} \subseteq \mathbb{R}$, $\sup_{\mathbf{x}\in\mathscr{X}} \mathbb{E}[|y_i|^{2+v}|\mathbf{x}_i = \mathbf{x}] \leq 2$ for some $v > 0$, and the following conditions hold.*

(i) *$\mathscr{G}$ and $\mathscr{H}$ are real-valued pointwise measurable classes of functions on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathbb{P}_X)$.*

(ii) *There exists a surrogate measure $\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}$ for $\mathbb{P}_X$ with respect to $\mathscr{G}\cup\mathscr{H}$ such that $\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}} = \mathfrak{m} \circ \phi_{\mathscr{G}\cup\mathscr{H}}$, where the normalizing transformation $\phi_{\mathscr{G}\cup\mathscr{H}} : \mathscr{Q}_{\mathscr{G}\cup\mathscr{H}} \to [0,1]^d$ is a diffeomorphism.*

(iii) *$\mathscr{G}$ is a VC-type class with envelope $\mathtt{M}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}$ over $\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}$ with $\mathtt{c}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}} \geq e$ and $\mathtt{d}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}} \geq 1$. $\mathscr{H}$ is a VC-type class with envelope $\mathtt{M}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}$ over $\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}$ with $\mathtt{c}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}} \geq e$ and $\mathtt{d}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}} \geq 1$.*

(iv) *$\mathscr{R}$ and $\mathscr{S}$ are real-valued pointwise measurable classes of functions on $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_Y)$.*

(v) *$\mathscr{R}$ is a VC-type class with envelope $M_{\mathscr{R},\mathscr{Y}}$ over $\mathscr{Y}$ with $\mathtt{c}_{\mathscr{R},\mathscr{Y}} \geq e$ and $\mathtt{d}_{\mathscr{R},\mathscr{Y}} \geq 1$, where $M_{\mathscr{R},\mathscr{Y}}(y) + \mathtt{pTV}_{\mathscr{R},(-|y|,|y|)} \leq \mathtt{v}(1+|y|)$ for all $y \in \mathscr{Y}$, for some $\mathtt{v} > 0$. $\mathscr{S}$ is a VC-type class with envelope $M_{\mathscr{S},\mathscr{Y}}$ over $\mathscr{Y}$ with $\mathtt{c}_{\mathscr{S},\mathscr{Y}} \geq e$ and $\mathtt{d}_{\mathscr{S},\mathscr{Y}} \geq 1$, where $M_{\mathscr{S},\mathscr{Y}}(y) + \mathtt{pTV}_{\mathscr{S},(-|y|,|y|)} \leq \mathtt{v}(1+|y|)$ for all $y \in \mathscr{Y}$, for some $\mathtt{v} > 0$.*

(vi) *There exists a constant $\mathtt{k}$ such that $|\log_2 \mathtt{E}| + |\log_2 \mathtt{TV}| + |\log_2 \mathtt{M}| + |\log_2 \mathtt{L}| \leq \mathtt{k}\log_2(n)$, where $\mathtt{E} = \max\{\mathtt{E}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}, \mathtt{E}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\}$, $\mathtt{TV} = \max\{\mathtt{TV}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}, \mathtt{TV}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\}$, $\mathtt{M} = \max\{\mathtt{M}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}, \mathtt{M}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\}$, and $\mathtt{L} = \max\{\mathtt{L}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}, \mathtt{L}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\}$.*

Consider the empirical process

$$A_n(g,h,r,s) = M_n(g,r) + M_n(h,s), \qquad g \in \mathscr{G}, r \in \mathscr{R}, h \in \mathscr{H}, s \in \mathscr{S}.$$

Then, on a possibly enlarged probability space, there exists a sequence of mean-zero Gaussian processes $(Z_n^A(g,h,r,s) : g \in \mathscr{G}, h \in \mathscr{H}, r \in \mathscr{R}, s \in \mathscr{S})$ with almost sure continuous trajectories such that:

- $\mathbb{E}[A_n(g_1,h_1,r_1,s_1)A_n(g_2,h_2,r_2,s_2)] = \mathbb{E}[Z_n^A(g_1,h_1,r_1,s_1)Z_n^A(g_2,h_2,r_2,s_2)]$ *holds for all (g_1,h_1,r_1,s_1), $(g_2,h_2,r_2,s_2) \in \mathscr{G}\times\mathscr{H}\times\mathscr{R}\times\mathscr{S}$, and*
- $\mathbb{E}\big[\,\|A_n - Z_n^A\|_{\mathscr{G}\times\mathscr{H}\times\mathscr{R}\times\mathscr{S}}\,\big] \leq C\mathtt{v}((\mathtt{d}\log(\mathtt{c}n))^{\frac{3}{2}}\mathtt{r}_n^{\frac{v}{v+2}}(\sqrt{\mathtt{ME}})^{\frac{2}{v+2}} + \mathtt{d}\log(\mathtt{c}n)\mathtt{M}n^{-\frac{v/2}{2+v}} + \mathtt{d}\log(\mathtt{c}n)\mathtt{M}n^{-\frac{1}{2}}\Big(\frac{\sqrt{\mathtt{ME}}}{\mathtt{r}_n}\Big)^{\frac{2}{v+2}})$,

where C is a universal constant, $\mathtt{c} = \mathtt{c}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}} + \mathtt{c}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}} + \mathtt{c}_{\mathscr{R},\mathscr{Y}} + \mathtt{c}_{\mathscr{S},\mathscr{Y}} + \mathtt{k}$, $\mathtt{d} = \mathtt{d}_{\mathscr{G},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\mathtt{d}_{\mathscr{H},\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\mathtt{d}_{\mathscr{R},\mathscr{Y}}\mathtt{d}_{\mathscr{S},\mathscr{Y}}\mathtt{k}$,

$$\mathtt{r}_n = \min\Big\{\frac{(\mathtt{c}_1^d\mathtt{M}^{d+1}\mathtt{TV}^d\mathtt{E})^{1/(2d+2)}}{n^{1/(2d+2)}}, \frac{(\mathtt{c}_1^{\frac{d}{2}}\mathtt{c}_2^{\frac{d}{2}}\mathtt{MTV}^{\frac{d}{2}}\mathtt{EL}^{\frac{d}{2}})^{1/(d+2)}}{n^{1/(d+2)}}\Big\},$$
$$\mathtt{c}_1 = d\sup_{\mathbf{x}\in\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\prod_{j=1}^{d-1}\sigma_j(\nabla\phi_{\mathscr{G}\cup\mathscr{H}}(\mathbf{x})), \qquad \mathtt{c}_2 = \sup_{\mathbf{x}\in\mathbb{Q}_{\mathscr{G}\cup\mathscr{H}}}\frac{1}{\sigma_d(\nabla\phi_{\mathscr{G}\cup\mathscr{H}}(\mathbf{x}))}.$$

SA-7.3 Proof of Theorem SA-25

Let $\mathscr{I}_A = \mathscr{G}\times\mathscr{H}\times\mathscr{R}\times\mathscr{S}$. For each $\iota = (g,h,r,s) \in \mathscr{I}_A$, define the combined summand

$$a_\iota(\mathbf{x},y) = g(\mathbf{x})r(y) + h(\mathbf{x})s(y), \qquad \mathscr{F}_A = \{a_\iota : \iota \in \mathscr{I}_A\}.$$

The proof below adapts the projection and coupling construction in Cattaneo and Yu [2025, Theorem SA.1]. The adaptation has two purposes. First, the two empirical processes cannot be coupled separately, because doing so would generally lose the covariance between $M_n(g,r)$ and $M_n(h,s)$. Second, a direct generic strong approximation for the ordinary class $\mathscr{F}_A$ would treat $(\mathbf{x},y)$ symmetrically and would therefore not exploit the multiplicative-separable structure. We instead use one common normalizing transformation, one common projection, and one common Gaussian construction for the whole sum a_ι.

Reduction and common normalization. Let $\widetilde{\mathscr{G}} = \{g\circ\phi^{-1}_{\mathscr{G}\cup\mathscr{H}} : g \in \mathscr{G}\}$ and $\widetilde{\mathscr{H}} = \{h\circ\phi^{-1}_{\mathscr{G}\cup\mathscr{H}} : h \in \mathscr{H}\}$. By the surrogate measure and normalizing-transformation argument used in the proof of Cattaneo and Yu

[2025, Theorem SA.1], after enlarging the probability space if necessary, we may work with i.i.d. variables $\mathbf{u}_i \sim \mathrm{Uniform}([0,1]^d)$ such that, simultaneously for all $g \in \mathscr{G}$ and $h \in \mathscr{H}$,

$$g(\mathbf{x}_i) = g \circ \phi^{-1}_{\mathscr{G}\cup\mathscr{H}}(\mathbf{u}_i), \qquad h(\mathbf{x}_i) = h \circ \phi^{-1}_{\mathscr{G}\cup\mathscr{H}}(\mathbf{u}_i).$$

This reduction is simultaneous for $\mathscr{G}$ and $\mathscr{H}$, which is why the surrogate measure is imposed on $\mathscr{G}\cup\mathscr{H}$. The constants $\mathtt{c}_1$ and $\mathtt{c}_2$ in the statement record the effect of this change of variables on the total-variation and Lipschitz constants; all displays below are written for the normalized variables and translated back through these constants.

Entropy and envelopes. The classes $\widetilde{\mathscr{G}}$ and $\widetilde{\mathscr{H}}$ remain pointwise measurable and VC-type after the change of variables. Products with $\mathscr{R}$ and $\mathscr{S}$ preserve VC-type entropy, and finite sums preserve the same property, so $\mathscr{F}_A$ is pointwise measurable and VC-type. For $\iota = (g, h, r, s)$,

$$|a_\iota(\mathbf{x}, y)| \le \mathtt{v}(1 + |y|)\big(M_{\mathscr{G},\mathscr{Q}_{\mathscr{G}\cup\mathscr{H}}}(\mathbf{x}) + M_{\mathscr{H},\mathscr{Q}_{\mathscr{G}\cup\mathscr{H}}}(\mathbf{x})\big).$$

Consequently the entropy constants for the combined index class are bounded by fixed powers of the constants appearing in the theorem; these powers are absorbed into $\mathtt{c}$ and $\mathtt{d}$.

Total Variation. The x-side L_1, envelope, Lipschitz, and total-variation quantities entering the projection argument are controlled by the corresponding maxima $\mathtt{E}$, $\mathtt{M}$, $\mathtt{L}$, and $\mathtt{TV}$. For example, for each fixed y and each cube $\mathscr{D} \subseteq [0,1]^d$,

$$\begin{aligned}
\mathtt{TV}_{\{a_\iota(\cdot,y):\iota\in\mathscr{I}_A\},\mathscr{D}} &\le \mathtt{v}(1 + |y|)\big(\mathtt{TV}_{\mathscr{G},\mathscr{D}} + \mathtt{TV}_{\mathscr{H},\mathscr{D}}\big),\\
\mathtt{L}_{\{a_\iota(\cdot,y):\iota\in\mathscr{I}_A\},\mathscr{D}} &\le \mathtt{v}(1 + |y|)\big(\mathtt{L}_{\mathscr{G},\mathscr{D}} + \mathtt{L}_{\mathscr{H},\mathscr{D}}\big).
\end{aligned}$$

The analogous pointwise total-variation bound in the y coordinate follows from Assumption (v):

$$\mathtt{pTV}_{\{a_\iota(\mathbf{x},\cdot):\iota\in\mathscr{I}_A\},(-|y|,|y|)} \le \mathtt{v}(1 + |y|)\big(M_{\mathscr{G},\mathscr{Q}_{\mathscr{G}\cup\mathscr{H}}}(\mathbf{x}) + M_{\mathscr{H},\mathscr{Q}_{\mathscr{G}\cup\mathscr{H}}}(\mathbf{x})\big).$$

Thus the meshing, projection, and Gaussian-projection constants in the proof of Cattaneo and Yu [2025, Theorem SA.1] are bounded by the same expressions as in that theorem, with $\mathscr{G}$ replaced by the worst case between $\mathscr{G}$ and $\mathscr{H}$ and with $\mathscr{R}$ replaced by the worst case between $\mathscr{R}$ and $\mathscr{S}$.

Common projection and coupling. Let π_δ denote a δ-net projection of $\mathscr{I}_A$ under the $L_2(\mathbb{P}_Z)$ semimetric induced by a_ι, and let Π denote the common cylindered Haar projection used in the proof of Cattaneo and Yu [2025, Theorem SA.1], with main depth in the normalized $\mathbf{u}$ coordinates and multiplier depth in the y coordinate. The same projected cells are used for the two summands $g(\mathbf{x})r(y)$ and $h(\mathbf{x})s(y)$. Therefore the projected empirical vector is the projection of the sum, not the sum of two independently coupled projections. For the Gaussian process Z_n^A generated by this common construction,

$$\begin{aligned}
\big\|A_n - Z_n^A\big\|_{\mathscr{I}_A} \le{}& \|A_n - A_n\circ\pi_\delta\|_{\mathscr{I}_A} + \|A_n\circ\pi_\delta - \Pi A_n\circ\pi_\delta\|_{\mathscr{I}_A}\\
&+ \big\|\Pi A_n\circ\pi_\delta - \Pi Z_n^A\circ\pi_\delta\big\|_{\mathscr{I}_A} + \big\|\Pi Z_n^A\circ\pi_\delta - Z_n^A\circ\pi_\delta\big\|_{\mathscr{I}_A} + \big\|Z_n^A\circ\pi_\delta - Z_n^A\big\|_{\mathscr{I}_A}.
\end{aligned}$$

The first and last terms are the empirical and Gaussian meshing errors. The second and fourth terms are the empirical and Gaussian projection errors. The middle term is the projected coupling error. Applying the same bounds as in the proof of Cattaneo and Yu [2025, Theorem SA.1] to this single projected vector,

and then optimizing over the quasi-dyadic projection depths, gives, in the bounded-transform version of Assumption (v),

$$\mathbb{E}\big[\,\big\|A_n - Z_n^A\big\|_{\mathscr{F}_A}\,\big] \lesssim \mathtt{v}(\mathtt{d}\log(\mathtt{c}n))^{3/2}\mathtt{r}_n.$$

The rate $\mathtt{r}_n$ is the minimum of the two projection/coupling balances displayed in the theorem. The first balance uses total variation; the second additionally uses the Lipschitz constants and yields the improvement available for Lipschitz classes.

Polynomial moments. It remains to remove the bounded-transform simplification. For $\eta > 0$, define truncated classes $r^\eta(y) = r(y)\mathbb{1}(|y| \le \eta)$ and $s^\eta(y) = s(y)\mathbb{1}(|y| \le \eta)$, and let A_n^η denote the corresponding process. By Assumption (v) and $\sup_{\mathbf{x}\in\mathscr{X}} \mathbb{E}[|y_i|^{2+v} \mid \mathbf{x}_i = \mathbf{x}] \le 2$,

$$\sup_{r\in\mathscr{R}}\sup_{\mathbf{x}\in\mathscr{X}} \mathbb{E}\big[|r(y_i)|^2\mathbb{1}(|y_i| > \eta) \mid \mathbf{x}_i = \mathbf{x}\big] \lesssim \mathtt{v}^2\eta^{-v},$$

and the same bound holds for $\mathscr{S}$. The empirical tail, the corresponding Gaussian tail, and the deterministic centering error are then controlled by Bernstein's inequality, Markov's inequality, and the preceding VC entropy bounds, giving

$$\begin{aligned}\mathbb{E}\big[\,\big\|A_n - Z_n^A\big\|_{\mathscr{F}_A}\,\big] \lesssim \mathtt{v}\Big\{&(\mathtt{d}\log(\mathtt{c}n))^{3/2}\big(\eta\mathtt{r}_n + \sqrt{\mathtt{ME}}\eta^{-v/2}\big)\\ &+ \mathtt{d}\log(\mathtt{c}n)\mathtt{M}n^{-\frac{v/2}{2+v}} + \mathtt{d}\log(\mathtt{c}n)\mathtt{M}n^{-1/2}\eta\Big\}.\end{aligned}$$

Choosing

$$\eta = \Big(\frac{\sqrt{\mathtt{ME}}}{\mathtt{r}_n}\Big)^{2/(v+2)}$$

balances the bounded-transform approximation error with the stochastic tail error, and yields exactly the bound in the theorem.

Finally, the common Gaussian construction is a Brownian-bridge construction indexed by the combined class $\mathscr{F}_A$. Hence, for any two indices $\iota_1, \iota_2 \in \mathscr{I}_A$,

$$\mathbb{E}[Z_n^A(\iota_1)Z_n^A(\iota_2)] = \mathbb{E}[A_n(\iota_1)A_n(\iota_2)].$$

This covariance identity includes the cross-covariance between $M_n(g, r)$ and $M_n(h, s)$ because the whole sum was projected and coupled jointly. Almost sure continuity of the sample paths follows from the same separability and semimetric-continuity argument used in Cattaneo and Yu [2025, Theorem SA.1], since the combined class is pointwise measurable and VC-type under the envelope above. This proves the theorem. □

SA-8 Proofs

SA-8.1 Proof of Lemma SA-1

For each j, the class $\{\mathbf{v} \in \mathscr{X} : f(\mathbf{v}) \ge 0,\ f \in \mathscr{V}_j\}$ is VC because $\mathscr{V}_j$ is finite dimensional. Fixed finite Boolean combinations of VC classes are VC; see, for example, the closure properties in van der Vaart and Wellner [1996, Section 2.6]. The displayed representation therefore implies that $\mathscr{D}_d$ is VC. If the balls are semialgebraic with uniformly bounded degree and Boolean complexity, the relevant polynomial inequalities belong to finite-dimensional spaces spanned by the monomials in $\mathbf{v}$ up to the maximal degree, and the same

argument applies. The listed distances are special cases. □

SA-8.2 Proof of Lemma SA-2

Let $\varsigma_t = 2t - 1$. On $\{\mathbf{X}_i \in \mathscr{A}_t\}$, $D_i(\mathbf{x}) = \varsigma_t\|\mathbf{X}_i - \mathbf{x}\|$. Assumption SA-1(ii) implies

$$\begin{aligned}\mathbf{\Psi}_{t,\mathbf{x}} &= \mathbb{E}\Big[\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{X}_i-\mathbf{x}\|}{h}\Big)\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{X}_i-\mathbf{x}\|}{h}\Big)^\top K_h(\varsigma_t\|\mathbf{X}_i-\mathbf{x}\|)\mathbb{1}(\mathbf{X}_i\in\mathscr{A}_t)\Big]\\ &= \int_{\mathscr{A}_t}\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{u}-\mathbf{x}\|}{h}\Big)\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{u}-\mathbf{x}\|}{h}\Big)^\top K_h(\varsigma_t\|\mathbf{u}-\mathbf{x}\|)f_X(\mathbf{u})\,\mathrm{d}\mathbf{u}\\ &= f_X(\mathbf{x})\int_{\mathscr{A}_t}\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{u}-\mathbf{x}\|}{h}\Big)\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{u}-\mathbf{x}\|}{h}\Big)^\top K_h(\varsigma_t\|\mathbf{u}-\mathbf{x}\|)\,\mathrm{d}\mathbf{u}+o(1),\end{aligned}$$

where in the last line we have used $\int_{\mathscr{A}_t}(\frac{\|\mathbf{u}-\mathbf{x}\|}{h})^v K_h(\varsigma_t\|\mathbf{u}-\mathbf{x}\|)\,\mathrm{d}\mathbf{u} = O(1)$ for each $0 \le v \le 2p$ from a standard change of variable argument.

I. Polynomial Representation of Minimum Eigenvalue

For simplicity, write

$$\mathbf{S}_{t,\mathbf{x}}(h) = \int_{\mathscr{A}_t}\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{u}-\mathbf{x}\|}{h}\Big)\mathbf{r}_p\Big(\frac{\varsigma_t\|\mathbf{u}-\mathbf{x}\|}{h}\Big)^\top K_h(\varsigma_t\|\mathbf{u}-\mathbf{x}\|)\,\mathrm{d}\mathbf{u}.$$

A change of variable gives

$$\mathbf{S}_{t,\mathbf{x}}(h) = \int \mathbf{r}_p(\varsigma_t\|\mathbf{z}\|)\mathbf{r}_p(\varsigma_t\|\mathbf{z}\|)^\top K(\varsigma_t\|\mathbf{z}\|)\mathbb{1}(\mathbf{x}+h\mathbf{z}\in\mathscr{A}_t)\,\mathrm{d}\mathbf{z}.$$

Let $\mathbf{a}\in\mathbb{R}^{p+1}$. Condition (c) in Lemma SA-2 implies the lower bound used below. If K is Lipschitz, then it is continuous, and positivity on the compact set U implies $\inf_{\mathbf{u}\in U}K(\varsigma_t\|\mathbf{u}\|) = \kappa > 0$. If $K(u) = \mathbb{1}(u\in[-1,1])$, positivity on U implies the same conclusion with $\kappa = 1$. Then the equivalent representation of minimum eigenvalue gives

$$\begin{aligned}\lambda_{\min}(\mathbf{S}_{t,\mathbf{x}}(h)) &= \min_{\|\mathbf{a}\|=1}\int(\mathbf{a}^\top\mathbf{r}_p(\varsigma_t\|\mathbf{z}\|))^2K(\varsigma_t\|\mathbf{z}\|)\mathbb{1}(\mathbf{x}+h\mathbf{z}\in\mathscr{A}_t)\,\mathrm{d}\mathbf{z}\\ &\ge \kappa\min_{\|\mathbf{a}\|=1}\int_U(\mathbf{a}^\top\mathbf{r}_p(\varsigma_t\|\mathbf{z}\|))^2\mathbb{1}(\mathbf{x}+h\mathbf{z}\in\mathscr{A}_t)\,\mathrm{d}\mathbf{z},\end{aligned}\tag{SA-11}$$

where the sign ς_t only premultiplies the polynomial basis by a nonsingular diagonal matrix, so the lower bound in terms of $\mathbf{r}_p(\|\mathbf{z}\|)$ below is equivalent up to fixed constants.

II. Mass Retaining Ratio in Treatment/Control Region

Denote $E_h(\mathbf{x},t) = \{\mathbf{z}\in U : \mathbf{x}+h\mathbf{z}\in\mathscr{A}_t\}$. Assumption SA-2(iii) implies there is some upper bound $\Lambda > 0$ of $K(\cdot)$. Hence for $c_0 = 1/2\ \liminf_{h\downarrow 0}\inf_{\mathbf{x}\in\mathscr{B}}\int_U K(\varsigma_t\|\mathbf{u}\|)\mathbb{1}(\mathbf{x}+h\mathbf{u}\in\mathscr{A}_t)\,\mathrm{d}\mathbf{u}$, we have

$$\Lambda\mathfrak{m}(E_h(\mathbf{x},t)) \ge \int_U K(\varsigma_t\|\mathbf{u}\|)\mathbb{1}(\mathbf{x}+h\mathbf{u}\in\mathscr{A}_t)\,\mathrm{d}\mathbf{u} \ge c_0$$

for small enough h, which implies

$$\mathfrak{m}(E_h(\mathbf{x},t)) \geq \alpha \mathfrak{m}(U), \qquad \alpha = \frac{c_0}{\Lambda \mathfrak{m}(U)}. \tag{SA-12}$$

III. L_2 Integral of Polynomials in Full Versus Treatment/Control Regions

Consider $S = \{q \in \mathscr{P}_p : \int_U q(\|\mathbf{u}\|)^2 \, \mathrm{d}\mathbf{u} = 1\}$, where $\mathscr{P}_p$ is the collection of all polynomials of degree at most p. Equip $\mathscr{P}_p$ with the inner product induced by $\langle q_1, q_2 \rangle_U = \int_U q_1(\|\mathbf{u}\|) q_2(\|\mathbf{u}\|) \, \mathrm{d}\mathbf{u}$. Let $(\phi_j, 1 \leq j \leq p+1)$ be an orthonormal basis for this finite-dimensional space. Then $T(\mathbf{a}) = \sum_{j=1}^{p+1} a_j \phi_j$ maps the Euclidean unit sphere in $\mathbb{R}^{p+1}$ onto S. Hence S is compact under this $L_2(U)$ norm, and finite-dimensional norm equivalence implies that S is also compact under the corresponding supremum norm on U. Now consider

$$\Phi_q(\varepsilon) = \mathfrak{m}(\{\mathbf{u} \in U : |q(\|\mathbf{u}\|)| < \varepsilon\}), \qquad q \in S, \varepsilon > 0,$$

and

$$\psi(q) = \sup\Big\{\varepsilon > 0 : \Phi_q(\varepsilon) \leq \frac{\alpha}{2}\mathfrak{m}(U)\Big\}.$$

Since $\int_U q(\|\mathbf{u}\|)^2 \, \mathrm{d}\mathbf{u} = 1$ and q is a polynomial, $\lim_{\varepsilon \downarrow 0} \Phi_q(\varepsilon) = 0$ and $\Phi_q(\varepsilon) = \mathfrak{m}(U)$ for all sufficiently large ε. Continuity of $q \in S$ implies $\psi(q) > 0$ for all $q \in S$.

Next, we want to show ψ is a lower-semicontinuous function on $(\mathscr{P}_p, \|\cdot\|_{L_\infty})$. Suppose $q_n \to q$ uniformly on U. For every $\varepsilon_0 \in (0, \psi(q))$, choose $\eta > \varepsilon_0$ such that $\Phi_q(\eta) \leq \frac{\alpha}{2}\mathfrak{m}(U)$. For large enough n, uniform convergence gives $|q_n(\|\mathbf{u}\|) - q(\|\mathbf{u}\|)| \leq \eta - \varepsilon_0$ on U, so $\{|q_n(\|\cdot\|)| < \varepsilon_0\} \subseteq \{|q(\|\cdot\|)| < \eta\}$. Hence $\Phi_{q_n}(\varepsilon_0) \leq \Phi_q(\eta) \leq \frac{\alpha}{2}\mathfrak{m}(U)$, which implies $\varepsilon_0 \leq \psi(q_n)$. This implies $\liminf_{n\to\infty} \psi(q_n) \geq \varepsilon_0$. Since ε_0 is arbitrary in $(0, \psi(q))$, we have $\liminf_{n\to\infty} \psi(q_n) \geq \psi(q)$.

Compactness of S and lower-semicontinuity of ψ implies ψ attains its minimum on S. Since $\psi(q) > 0$ for all $q \in S$, we know $\varepsilon_* = \inf_{q \in S} \psi(q) > 0$. Then for every $q \in S$,

$$\begin{aligned}
\int_{E_h(\mathbf{x},t)} q(\|\mathbf{u}\|)^2 \, \mathrm{d}\mathbf{u} &\geq \Big(\frac{\varepsilon_*}{2}\Big)^2 \mathfrak{m}\Big(E_h(\mathbf{x},t) \setminus \{\mathbf{u} \in U : |q(\|\mathbf{u}\|)| \leq \varepsilon_*/2\}\Big) \\
&\geq \Big(\frac{\varepsilon_*}{2}\Big)^2 \Big(\mathfrak{m}(E_h(\mathbf{x},t)) - \mathfrak{m}(\{\mathbf{u} \in U : |q(\|\mathbf{u}\|)| \leq \varepsilon_*/2\})\Big) \\
&\geq \frac{\varepsilon_*^2 \alpha}{8}\mathfrak{m}(U).
\end{aligned}$$

Scaling q from S gives

$$\int_{E_h(\mathbf{x},t)} q(\|\mathbf{u}\|)^2 \, \mathrm{d}\mathbf{u} \geq \frac{\varepsilon_*^2 \alpha}{8} \int_U q(\|\mathbf{u}\|)^2 \, \mathrm{d}\mathbf{u}, \qquad q \in \mathscr{P}_p. \tag{SA-13}$$

IV. Lower Bound of Minimum Eigenvalue

Equations (SA-11), (SA-12) and (SA-13) together give for small enough h,

$$\begin{aligned}
\inf_{\mathbf{x} \in \mathscr{B}} \lambda_{\min}(\mathbf{S}_{t,\mathbf{x}}(h)) &\geq \kappa \inf_{\mathbf{x} \in \mathscr{B}} \min_{\|\mathbf{a}\|=1} \int_{E_h(\mathbf{x},t)} (\mathbf{a}^\top \mathbf{r}_p(\|\mathbf{z}\|))^2 \, \mathrm{d}\mathbf{z} \\
&\geq \kappa \frac{\varepsilon_*^2 \alpha}{8} \min_{\|\mathbf{a}\|=1} \int_U (\mathbf{a}^\top \mathbf{r}_p(\|\mathbf{z}\|))^2 \, \mathrm{d}\mathbf{z}
\end{aligned}$$

$$\geq \kappa \frac{\varepsilon_*^2 \alpha}{8} \lambda_{\min}\Big(\int_U \mathbf{r}_p(\|\mathbf{z}\|)\mathbf{r}_p(\|\mathbf{z}\|)^\top \, \mathrm{d}\mathbf{z}\Big),$$

which implies $\liminf_{h\to 0} \inf_{\mathbf{x}\in\mathscr{B}} \lambda_{\min}(\mathbf{S}_{t,\mathbf{x}}(h)) > 0$. Because f_X is continuous and bounded away from zero on $\mathscr{X}$ and $\mathbf{\Psi}_{t,\mathbf{x}} = f_X(\mathbf{x})\mathbf{S}_{t,\mathbf{x}}(h) + o(1)$ uniformly in $\mathbf{x} \in \mathscr{B}$, the same lower bound holds for $\mathbf{\Psi}_{t,\mathbf{x}}$. This proves Assumption SA-2(iv). □

SA-8.3 Proof of Lemma SA-3

Since $\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}$ is a finite dimensional matrix, it suffices to show the stated rate of convergence for each entry. For $0 \leq v \leq 2p$, define $\mathscr{G} = \{g_{n,t,v}(\cdot,\mathbf{x}) : \mathbf{x} \in \mathscr{X}\}$ with

$$g_{n,t,v}(\xi,\mathbf{x}) = \Big(\frac{\varsigma_t \mathscr{d}(\xi,\mathbf{x})}{h}\Big)^v \frac{1}{h^d} K\Big(\frac{\varsigma_t \mathscr{d}(\xi,\mathbf{x})}{h}\Big) \mathbb{1}(\xi \in \mathscr{A}_t), \qquad \xi,\mathbf{x} \in \mathscr{X}.$$

These are the signed-distance matrix entries, up to the finite choice of v. We will show $\mathscr{G}$ is a VC-type class.

Constant Envelope Function. Since K is bounded and compactly supported, there exists a constant C_1 such that $\sup_{\mathbf{x}\in\mathscr{X}} \|g_{n,t,v}(\cdot,\mathbf{x})\|_\infty \leq C_1 h^{-d} = G$.

Diameter of $\mathscr{G}$ in L_2. Let $R_K < \infty$ satisfy $\mathrm{supp}(K) \subseteq [-R_K, R_K]$. For each $\mathbf{x} \in \mathscr{X}$, $g_{n,t,v}(\cdot,\mathbf{x})$ is supported on $\{\xi : \mathscr{d}(\xi,\mathbf{x}) \leq R_K h\}$. By Assumption SA-1(ii) and Assumption SA-2(i), $\sup_{\mathbf{x}\in\mathscr{X}} \mathbb{P}\left(\mathscr{d}(\mathbf{X}_i,\mathbf{x}) \leq R_K h\right) \lesssim h^d$. It follows that $\sup_{\mathbf{x}\in\mathscr{X}} \|g_{n,t,v}(\cdot,\mathbf{x})\|_{\mathbb{P},2} \leq C_2 h^{-d/2}$ for some constant C_2. We can take C_1 large enough so that $\sigma = C_2 h^{-d/2} \leq G = C_1 h^{-d}$.

Ratio. For some constant C_3, $\delta = \frac{\sigma}{F} = C_3\sqrt{h^d}$.

Covering Numbers. Case 1: K is Lipschitz. Let $\mathbf{x},\mathbf{x}' \in \mathscr{X}$. By Assumption SA-2,

$$\begin{aligned}
&\sup_{\xi\in\mathscr{X}} \left| g_{n,t,v}(\xi,\mathbf{x}) - g_{n,t,v}(\xi,\mathbf{x}')\right| \\
&\leq \sup_{\xi\in\mathscr{X}} \left| \Big[\Big(\frac{\varsigma_t \mathscr{d}(\xi,\mathbf{x})}{h}\Big)^v - \Big(\frac{\varsigma_t \mathscr{d}(\xi,\mathbf{x}')}{h}\Big)^v\Big] K_h(\varsigma_t \mathscr{d}(\xi,\mathbf{x}))\right| \mathbb{1}(\xi \in \mathscr{A}_t) \\
&\quad + \left| \Big(\frac{\varsigma_t \mathscr{d}(\xi,\mathbf{x}')}{h}\Big)^v \Big[K_h(\varsigma_t \mathscr{d}(\xi,\mathbf{x})) - K_h(\varsigma_t \mathscr{d}(\xi,\mathbf{x}'))\Big]\right| \mathbb{1}(\xi \in \mathscr{A}_t) \\
&\lesssim h^{-d-1} \|\mathbf{x} - \mathbf{x}'\|_\infty.
\end{aligned}$$

By Lipschitz continuity property of $\mathscr{G}$, for any $\varepsilon \in (0,1]$ and for any finitely supported measure Q and metric $\|\cdot\|_{Q,2}$ based on $L_2(Q)$,

$$N(\{g_{n,t,v}(\cdot,\mathbf{x}) : \mathbf{x} \in \mathscr{X}\}, \|\cdot\|_{Q,2}, \varepsilon\|G\|_{Q,2}) \leq N(\mathscr{X}, \|\cdot\|_\infty, \varepsilon\|G\|_{Q,2} h^{d+1}) \overset{(i)}{\lesssim} \Big(\frac{\mathrm{diam}(\mathscr{X})}{\varepsilon\|G\|_{Q,2} h^{d+1}}\Big)^d \lesssim \Big(\frac{\mathrm{diam}(\mathscr{X})}{\varepsilon h}\Big)^d,$$

where inequality (i) uses the fact that $\varepsilon\|G\|_{Q,2} h^{d+1} \lesssim \varepsilon h \lesssim 1$. Therefore, for constants C_1 and C_2 independent of Q,

$$\sup_Q N(\mathscr{G}, \|\cdot\|_{Q,2}, \varepsilon \|G\|_{Q,2}) \leq (C_1/\varepsilon)^{C_2}, \qquad \varepsilon \in (0,1],$$

where the supremum is taken over all finite discrete measures on $\mathscr{X}$.

Case 2: $K = \mathbb{1}(\cdot \in [-1,1])$. Consider

$$m_n(\xi, \mathbf{x}) = \Big(\frac{\varsigma_t \mathscr{d}(\xi, \mathbf{x})}{h}\Big)^v \frac{1}{h^d} \mathbb{1}(\xi \in \mathscr{A}_t) \mathbb{1}(\mathscr{d}(\xi, \mathbf{x}) \le h), \qquad \xi, \mathbf{x} \in \mathscr{X},$$

$\mathscr{M} = \{m_n(\cdot, \mathbf{x}) : \mathbf{x} \in \mathscr{B}\}$ and the constant envelope function $M = C_4 h^{-d}$, for some constant C_4 depending only on the diameter of $\mathscr{X}$. The polynomial factor is covered by the preceding Lipschitz argument. The class $\mathscr{L} = \{\mathbb{1}(\mathscr{d}(\cdot, \mathbf{x}) \le h) : \mathbf{x} \in \mathscr{B}\}$ has finite VC dimension by Assumption SA-2(iii) (for the Euclidean distance, see van der Vaart and Wellner, 1996, Example 2.6.1); products with the fixed assignment indicator preserve VC-type entropy. Hence, by van der Vaart and Wellner [1996, Theorem 2.6.4],

$$\sup_Q N(\mathscr{G}, \|\cdot\|_{Q,2}, \varepsilon \|G\|_{Q,2}) \le (C_1/\varepsilon)^{C_2}, \qquad \varepsilon \in (0,1],$$

where the supremum is taken over all finite discrete measures on $\mathscr{X}$.

Maximal Inequality. By Chernozhukov et al. [2014b, Corollary 5.1] for the empirical process on class $\mathscr{G}$,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{l \in \mathscr{G}} \big|\mathbb{E}_n\left[l(\mathbf{X}_i)\right] - \mathbb{E}[l(\mathbf{X}_i)]\big|\Big] &\lesssim \frac{\sigma}{\sqrt{n}} \sqrt{C_2 \log(C_1/\delta)} + \frac{\|G\|_{\mathbb{P},2} C_2 \log(C_1/\delta)}{n} \\
&\lesssim \frac{1}{\sqrt{nh^d}} \sqrt{d \log\Big(\frac{\operatorname{diam}(\mathscr{X})}{h^{1+d/2}}\Big)} + \frac{1}{nh^d} d \log\Big(\frac{\operatorname{diam}(\mathscr{X})}{h^{1+d/2}}\Big) \\
&\lesssim \sqrt{\frac{\log(1/h)}{nh^d}}.
\end{aligned}$$

Thus, $\sup_{\mathbf{x} \in \mathscr{B}} \big\|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}} - \mathbf{\Psi}_{t,\mathbf{x}}\big\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}}$.

By Weyl's Theorem, $\sup_{\mathbf{x} \in \mathscr{B}} |\lambda_{\min}(\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}) - \lambda_{\min}(\mathbf{\Psi}_{t,\mathbf{x}})| \le \sup_{\mathbf{x} \in \mathscr{B}} \|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}} - \mathbf{\Psi}_{t,\mathbf{x}}\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}}$. Therefore, we can lower bound the minimum eigenvalue by $\inf_{\mathbf{x} \in \mathscr{B}} \lambda_{\min}(\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}) \ge \inf_{\mathbf{x} \in \mathscr{B}} \lambda_{\min}(\mathbf{\Psi}_{t,\mathbf{x}}) - \sup_{\mathbf{x} \in \mathscr{B}} |\lambda_{\min}(\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}) - \lambda_{\min}(\mathbf{\Psi}_{t,\mathbf{x}})| \gtrsim_{\mathbb{P}} 1$.

Finally, it follows that $\sup_{\mathbf{x} \in \mathscr{B}} \|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1}\| \lesssim_{\mathbb{P}} 1$ and hence

$$\sup_{\mathbf{x} \in \mathscr{B}} \big\|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1} - \mathbf{\Psi}_{t,\mathbf{x}}^{-1}\big\| \le \sup_{\mathbf{x} \in \mathscr{B}} \big\|\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\big\| \big\|\mathbf{\Psi}_{t,\mathbf{x}} - \widehat{\mathbf{\Psi}}_{t,\mathbf{x}}\big\| \big\|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1}\big\| \lesssim_{\mathbb{P}} \sqrt{\frac{\log(1/h)}{nh^d}},$$

which completes the proof. □

SA-8.4 Proof of Lemma SA-4

Consider the class $\mathscr{F} = \{(\mathbf{z}, u) \mapsto \mathbf{e}_\ell^\top g_{t,\mathbf{x}}(\mathbf{z})(u - h_{\mathbf{x}}(\mathbf{z})) : \mathbf{x} \in \mathscr{B}\}$, $0 \le \ell \le p$, where for $\mathbf{z} \in \mathscr{X}$,

$$g_{t,\mathbf{x}}(\mathbf{z}) = \mathbf{r}_p\Big(\frac{D_{\mathbf{z}}(\mathbf{x})}{h}\Big) K_h(D_{\mathbf{z}}(\mathbf{x})) \mathbb{1}(D_{\mathbf{z}}(\mathbf{x}) \in \mathscr{I}_t), \qquad h_{\mathbf{x}}(\mathbf{z}) = \boldsymbol{\gamma}_t^*(\mathbf{x})^\top \mathbf{r}_p\left(D_{\mathbf{z}}(\mathbf{x})/h\right).$$

By definition of $\boldsymbol{\gamma}_t^*(\mathbf{x})$,

$$\boldsymbol{\gamma}_t^*(\mathbf{x}) = \mathbf{\Psi}_{t,\mathbf{x}}^{-1} \mathbf{S}_{t,\mathbf{x}}, \qquad \mathbf{S}_{t,\mathbf{x}} = \mathbb{E}\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big) K_h(D_i(\mathbf{x})) Y_i \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_t)\Big]. \tag{SA-14}$$

Assumption SA-1 implies $\mathbf{S}_{t,\mathbf{x}}$ is continuous in $\mathbf{x}$, hence $\sup_{\mathbf{x}\in\mathscr{B}}\|\mathbf{S}_{t,\mathbf{x}}\|\lesssim 1$. And by Assumption SA-2(iv), $\inf_{\mathbf{x}\in\mathscr{B}}\lambda_{\min}(\mathbf{\Psi}_{t,\mathbf{x}})\gtrsim 1$. Hence

$$\sup_{\mathbf{x}\in\mathscr{B}}\left\|\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{S}_{t,\mathbf{x}}\right\|\lesssim 1. \tag{SA-15}$$

Now, consider properties of $\mathscr{F}$. Definition of $\boldsymbol{\gamma}_t^*(\mathbf{x})$ implies $\mathbb{E}[f(\mathbf{X}_i,Y_i)]=0$ for all $f\in\mathscr{F}$. Since K is compactly supported, there exists $C_1,C_2>0$ such that $F(\mathbf{z},u)=C_1h^{-d}(|u|+C_2)$ is an envelope function for $\mathscr{F}$. Denote $M=\max_{1\le i\le n}F(\mathbf{X}_i,Y_i)$, then

$$\begin{aligned}\mathbb{E}[M^2]^{1/2}&\lesssim h^{-d}\mathbb{E}\left[\max_{1\le i\le n}|Y_i|^2+1\right]^{1/2}\lesssim h^{-d}\mathbb{E}\left[\max_{1\le i\le n}|Y_i|^{2+v}\right]^{1/(2+v)}\\&\lesssim h^{-d}\left[\sum_{i=1}^n\mathbb{E}[|Y_i|^{2+v}]\right]^{1/(2+v)}\lesssim h^{-d}n^{1/(2+v)},\end{aligned}$$

where we have used the uniform $(2+v)$ moment bound in Assumption SA-1(v). Denote $\sigma=\sup_{f\in\mathscr{F}}\mathbb{E}[f(\mathbf{X}_i,Y_i)^2]^{1/2}$. Then,

$$\sigma^2\lesssim\sup_{\mathbf{x}\in\mathscr{B}}\mathbb{E}[\|\mathbf{e}_\ell^\top g_{t,\mathbf{x}}\|_\infty^2(|Y_i|+\|h_{\mathbf{x}}\|_\infty)^2]\lesssim h^{-d}.$$

To check the covering number of $\mathscr{F}$, note that compared to the proof of Lemma SA-3, we have one additional term $\mathbf{e}_\ell^\top g_{t,\mathbf{x}}h_{\mathbf{x}}$. All signed-distance kernel and indicator terms can be handled as in the proof of Lemma SA-3. Recall Equation (SA-14), and consider $l_{t,\mathbf{x}}(\cdot)=\mathbf{e}_\ell^\top\mathbf{r}_p(D_\cdot(\mathbf{x})/h)K_h(D_\cdot(\mathbf{x}))\mu_t(\cdot)\mathbb{1}(D_\cdot(\mathbf{x})\in\mathscr{I}_t)$ and $\mathscr{L}_t=\{l_{t,\mathbf{x}}:\mathbf{x}\in\mathscr{B}\}$. Then, for any $\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}$,

$$|\mathbf{S}_{t,\mathbf{x}_1}-\mathbf{S}_{t,\mathbf{x}_2}|\le\|l_{t,\mathbf{x}_1}-l_{t,\mathbf{x}_2}\|_{\mathbb{P}_X,2},$$

and hence

$$N(\{\mathbf{e}_\ell^\top\mathbf{S}_{t,\mathbf{x}}:\mathbf{x}\in\mathscr{B}\},|\cdot|,\varepsilon h^{-d})\le N(\mathscr{L}_t,\|\cdot\|_{\mathbb{P}_X,2},\varepsilon h^{-d})\le\sup_Q N(\mathscr{L}_t,\|\cdot\|_{Q,2},\varepsilon h^{-d}),$$

The same argument as in the paragraph **Covering Numbers** in the proof of Lemma SA-3 then shows

$$\begin{aligned}\sup_Q N(\{g_{t,\mathbf{x}}:\mathbf{x}\in\mathscr{B}\},\|\cdot\|_{Q,2},\varepsilon C_1h^{-d})&\le\Big(\frac{\operatorname{diam}(\mathcal{X})}{h\varepsilon}\Big)^d,\qquad 0<\varepsilon\le 1,\\\sup_Q N(\{g_{t,\mathbf{x}}h_{\mathbf{x}}:\mathbf{x}\in\mathscr{B}\},\|\cdot\|_{Q,2},\varepsilon C_1h^{-d})&\le\Big(\frac{\operatorname{diam}(\mathcal{X})}{h\varepsilon}\Big)^d,\qquad 0<\varepsilon\le 1,\end{aligned}$$

where sup is taken over all discrete measures on $\mathcal{X}$. Product $\{g_{t,\mathbf{x}}:\mathbf{x}\in\mathscr{B}\}$ with the singleton identity class and addition of $\{g_{t,\mathbf{x}}h_{\mathbf{x}}:\mathbf{x}\in\mathscr{B}\}$ preserve VC-type entropy, so

$$\sup_Q N(\mathscr{F},\|\cdot\|_{Q,2},\varepsilon\|F\|_{Q,2})\le 2\Big(\frac{2\operatorname{diam}(\mathcal{X})}{h\varepsilon}\Big)^d,\qquad 0<\varepsilon\le 1,$$

where sup is taken over all discrete measures on $\mathcal{X}\times\mathbb{R}$. Denote $\mathtt{C}_1=d$, $\mathtt{C}_2=\frac{2(2\operatorname{diam}(\mathcal{X}))^d}{h^d}$. Hence, by

Chernozhukov et al. [2014b, Corollary 5.1]

$$\begin{aligned}\mathbb{E}\left[\sup_{\mathbf{x}\in\mathscr{B}}|\mathbf{e}_\ell^\top \mathbf{O}_{t,\mathbf{x}}|\right] &= \mathbb{E}\left[\sup_{f\in\mathscr{F}}|\mathbb{E}_n\left[f(\mathbf{X}_i,Y_i)\right]-\mathbb{E}[f(\mathbf{X}_i,Y_i)]|\right]\\ &\lesssim \frac{\sigma}{\sqrt{n}}\sqrt{\mathtt{C}_2\log(\mathtt{C}_1\left\|M\right\|_{\mathbb{P},2}/\sigma)}+\frac{\left\|M\right\|_{\mathbb{P},2}\mathtt{C}_2\log(\mathtt{C}_1\left\|M\right\|_{\mathbb{P},2}/\sigma)}{n}\\ &\lesssim \frac{1}{\sqrt{nh^d}}\sqrt{d\log\left(\frac{\operatorname{diam}(\mathscr{X})}{h^{1+d/2}}\right)}+\frac{1}{n^{\frac{1+v}{2+v}}h^d}d\log\left(\frac{\operatorname{diam}(\mathscr{X})}{h^{1+d/2}}\right)\\ &\lesssim \sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d}.\end{aligned}$$

The rest follows from finite dimensionality of $\mathbf{O}_{t,\mathbf{x}}$, and Lemma SA-3. □

SA-8.5 Proof of Lemma SA-5

Denote, for $\mathbf{z}\in\mathscr{B}$, $\eta_{i,t,\mathbf{z}}=Y_i-\theta^*_{t,\mathbf{z}}(D_i(\mathbf{z}))$ and $\xi_{i,t,\mathbf{z}}=\theta^*_{t,\mathbf{z}}(D_i(\mathbf{z}))-\widehat{\theta}_{t,\mathbf{z}}(D_i(\mathbf{z}))$. Then

$$\begin{aligned}\widehat{\boldsymbol{\Upsilon}}_{t,\mathbf{x},\mathbf{y}} = \mathbb{E}_n\Bigg[&\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\Big(\frac{D_i(\mathbf{y})}{h}\Big)^\top h^d K_h(D_i(\mathbf{x}))K_h(D_i(\mathbf{y}))\\ &\cdot(\eta_{i,t,\mathbf{x}}+\xi_{i,t,\mathbf{x}})(\eta_{i,t,\mathbf{y}}+\xi_{i,t,\mathbf{y}})\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\mathbb{1}(D_i(\mathbf{y})\in\mathscr{I}_t)\Bigg],\end{aligned}$$

and we decompose the error into

$$\begin{aligned}&\widehat{\boldsymbol{\Upsilon}}_{t,\mathbf{x},\mathbf{y}}-\boldsymbol{\Upsilon}_{t,\mathbf{x},\mathbf{y}}=\Delta_{1,t,\mathbf{x},\mathbf{y}}+\Delta_{2,t,\mathbf{x},\mathbf{y}}+\Delta_{3,t,\mathbf{x},\mathbf{y}},\\ &\Delta_{1,t,\mathbf{x},\mathbf{y}}=\mathbb{E}_n\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\left(\frac{D_i(\mathbf{y})}{h}\right)^\top h^d K_h(D_i(\mathbf{x}))K_h\left(D_i(\mathbf{y})\right)\xi_{i,t,\mathbf{x}}\xi_{i,t,\mathbf{y}}\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\mathbb{1}(D_i(\mathbf{y})\in\mathscr{I}_t)\right],\\ &\Delta_{2,t,\mathbf{x},\mathbf{y}}=\mathbb{E}_n\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\left(\frac{D_i(\mathbf{y})}{h}\right)^\top h^d K_h(D_i(\mathbf{x}))K_h\left(D_i(\mathbf{y})\right)(\eta_{i,t,\mathbf{x}}\xi_{i,t,\mathbf{y}}+\eta_{i,t,\mathbf{y}}\xi_{i,t,\mathbf{x}})\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\mathbb{1}(D_i(\mathbf{y})\in\mathscr{I}_t)\right],\\ &\Delta_{3,t,\mathbf{x},\mathbf{y}}=\mathbb{E}_n\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\left(\frac{D_i(\mathbf{y})}{h}\right)^\top h^d K_h(D_i(\mathbf{x}))K_h(D_i(\mathbf{y}))\eta_{i,t,\mathbf{x}}\eta_{i,t,\mathbf{y}}\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\mathbb{1}(D_i(\mathbf{y})\in\mathscr{I}_t)\right]\\ &\qquad-\mathbb{E}\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\left(\frac{D_i(\mathbf{y})}{h}\right)^\top h^d K_h(D_i(\mathbf{x}))K_h(D_i(\mathbf{y}))\eta_{i,t,\mathbf{x}}\eta_{i,t,\mathbf{y}}\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\mathbb{1}(D_i(\mathbf{y})\in\mathscr{I}_t)\right].\end{aligned}$$

By Assumption SA-2, $K_h(D_i(\mathbf{x}))\neq 0$ implies $\left\|\mathbf{r}_p(D_i(\mathbf{x})/h)\right\|_2\lesssim 1$. Hence by Lemma SA-3 and SA-4,

$$\begin{aligned}&\max_{t\in\{0,1\}}\max_{1\le i\le n}\sup_{\mathbf{x}\in\mathscr{B}}|\xi_{i,t,\mathbf{x}}|\\ &=\max_{t\in\{0,1\}}\max_{1\le i\le n}\sup_{\mathbf{x}\in\mathscr{B}}|\mathbf{r}_p(D_i(\mathbf{x})/h)^\top(\widehat{\boldsymbol{\gamma}}_{t,\mathbf{x}}-\boldsymbol{\gamma}^*_{t,\mathbf{x}})|\mathbb{1}(K_h(D_i(\mathbf{x}))\neq 0)\\ &=\max_{t\in\{0,1\}}\max_{1\le i\le n}\sup_{\mathbf{x}\in\mathscr{B}}|\mathbf{r}_p(D_i(\mathbf{x})/h)^\top\widehat{\boldsymbol{\Psi}}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}|\mathbb{1}(K_h(D_i(\mathbf{x}))\neq 0)\\ &\le\max_{t\in\{0,1\}}\sup_{\mathbf{x}\in\mathscr{B}}\left\|\widehat{\boldsymbol{\Psi}}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}\right\|_2\\ &\lesssim\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d},\end{aligned}$$

Under the rate conditions of the lemma, a maximal inequality similar to the one used in the proof of Lemma SA-3 shows

$$\sup_{\mathbf{x},\mathbf{y}\in\mathscr{B}} \|\Delta_{1,t,\mathbf{x},\mathbf{y}}\| \lesssim_{\mathbb{P}} \max_{t\in\{0,1\}} \max_{1\le i\le n} \sup_{\mathbf{x}\in\mathscr{B}} |\xi_{i,t,\mathbf{x}}|^2 \lesssim \left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d}\right)^2,$$
$$\sup_{\mathbf{x},\mathbf{y}\in\mathscr{B}} \|\Delta_{2,t,\mathbf{x},\mathbf{y}}\| \lesssim_{\mathbb{P}} \max_{t\in\{0,1\}} \max_{1\le i\le n} \sup_{\mathbf{x}\in\mathscr{B}} |\xi_{i,t,\mathbf{x}}| \lesssim \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d}. \tag{SA-16}$$

Consider the (μ,ν) entry of $\Delta_{3,t,\mathbf{x},\mathbf{y}}$ and the class

$$\mathscr{F} = \Big\{(\mathbf{z},u)\mapsto \Big(\frac{D_{\mathbf{z}}(\mathbf{x})}{h}\Big)^{\mu}\Big(\frac{D_{\mathbf{z}}(\mathbf{y})}{h}\Big)^{\nu} h^d K_h(D_{\mathbf{z}}(\mathbf{x}))K_h(D_{\mathbf{z}}(\mathbf{y}))$$
$$\times \big(u-\mathbf{r}_p(D_{\mathbf{z}}(\mathbf{x})/h)^\top\boldsymbol{\gamma}^*_{t,\mathbf{x}}\big)\big(u-\mathbf{r}_p(D_{\mathbf{z}}(\mathbf{y})/h)^\top\boldsymbol{\gamma}^*_{t,\mathbf{y}}\big)\mathbb{1}(D_{\mathbf{z}}(\mathbf{x})\in\mathscr{I}_t)\mathbb{1}(D_{\mathbf{z}}(\mathbf{y})\in\mathscr{I}_t) : \mathbf{x},\mathbf{y}\in\mathscr{B}\Big\}.$$

The kernel factors, side indicators, and polynomial factors form VC-type classes by the same covering arguments used above; finite products and sums preserve VC-type entropy with constants changing only by fixed powers. By Assumptions SA-2 and SA-1(v), we have $\sup_{f\in\mathscr{F}}\mathbb{E}[f(\mathbf{X}_i,Y_i)^2]^{1/2}\lesssim h^{-d/2}$. Moreover, Assumption SA-2 and Equation (SA-15) imply there exists $C_1,C_2>0$ such that $F(\mathbf{z},u)=C_1h^{-d}(u^2+C_2)$ is an envelope function for $\mathscr{F}$, with

$$\mathbb{E}\big[\max_{1\le i\le n}F(\mathbf{X}_i,Y_i)^2\big]^{\frac12}\lesssim C_1h^{-d}(\mathbb{E}\big[\max_{1\le i\le n}Y_i^4\big]^{\frac12}+C_2)\lesssim C_1h^{-d}(\mathbb{E}\big[\max_{1\le i\le n}Y_i^{2+v}\big]^{\frac{2}{2+v}}+C_2)\lesssim h^{-d}n^{\frac{2}{2+v}}.$$

Apply Chernozhukov et al. [2014b, Corollary 5.1] similarly as in Lemma SA-4 gives

$$\mathbb{E}\left[\sup_{f\in\mathscr{F}}|\mathbb{E}_n[f(\mathbf{X}_i,Y_i)]-\mathbb{E}[f(\mathbf{X}_i,Y_i)]|\right]\lesssim\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}.$$

Finite dimensionality of $\Delta_{3,t,\mathbf{x},\mathbf{y}}$ then implies

$$\mathbb{E}\left[\sup_{\mathbf{x},\mathbf{y}\in\mathscr{B}}\|\Delta_{3,t,\mathbf{x},\mathbf{y}}\|\right]\lesssim\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}. \tag{SA-17}$$

Putting together Equations (SA-16), (SA-17) and Lemma SA-3 gives the bound for $\widehat{\boldsymbol{\Upsilon}}_{t,\mathbf{x},\mathbf{y}}-\boldsymbol{\Upsilon}_{t,\mathbf{x},\mathbf{y}}$. The displayed bound for $\widehat{\Xi}-\Xi$ follows by pre- and post-multiplying by $\mathbf{e}_0^\top\widehat{\boldsymbol{\Psi}}^{-1}$ and its population analogue, and using the level definition of Ξ.

It remains to verify the scale of the variance term. The population lower bound $\lambda_{\min}(\boldsymbol{\Upsilon}_{t,\mathbf{x},\mathbf{x}})\gtrsim 1$ follows from the Gram lower bound and the conditional variance lower bound in Assumption SA-1(iv). To see why the replacement of K by K^2 is harmless, let $q_{\mathbf{a}}(u)=\mathbf{a}^\top\mathbf{r}_p(u)$ with $\|\mathbf{a}\|=1$. Assumption SA-2(iv) gives a uniform lower bound on the corresponding K-weighted local second moment. Since K is fixed, bounded, and compactly supported, while $q_{\mathbf{a}}$ is uniformly bounded on $\mathrm{supp}(K)$, choose $\delta>0$ small enough that the contribution from $\{K\le\delta\}$ is less than one half of this lower bound. On $\{K>\delta\}$, $K^2\ge\delta K$, which gives the same lower bound, up to constants, for the K^2-weighted matrix. Combining this with

$$\mathbb{E}\{(Y_i-\theta^*_{t,\mathbf{x}}(D_i(\mathbf{x})))^2\mid\mathbf{X}_i\}\ge\sigma_t^2(\mathbf{X}_i)$$

gives $\lambda_{\min}(\boldsymbol{\Upsilon}_{t,\mathbf{x},\mathbf{x}})\gtrsim 1$. The same local-moment calculation also gives $\sup_{\mathbf{x}\in\mathscr{B}}\|\boldsymbol{\Upsilon}_{t,\mathbf{x},\mathbf{x}}\|\lesssim 1$ and $\sup_{\mathbf{x}\in\mathscr{B}}\|\boldsymbol{\Psi}_{t,\mathbf{x}}\|\lesssim$

1. Therefore, uniformly in $\mathbf{x} \in \mathscr{B}$,

$$nh^d \Xi_{t,\mathbf{x},\mathbf{x}} = \left(\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{e}_0\right)^\top \mathbf{\Upsilon}_{t,\mathbf{x},\mathbf{x}} \left(\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{e}_0\right) \lesssim \|\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\|^2 \|\mathbf{\Upsilon}_{t,\mathbf{x},\mathbf{x}}\| \lesssim 1,$$

while

$$nh^d \Xi_{t,\mathbf{x},\mathbf{x}} \geq \lambda_{\min}(\mathbf{\Upsilon}_{t,\mathbf{x},\mathbf{x}}) \|\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{e}_0\|^2 \gtrsim \|\mathbf{\Psi}_{t,\mathbf{x}}\|^{-2} \gtrsim 1.$$

Summing over $t = 0, 1$ gives the same bound for $\Xi_{\mathbf{x}} = \Xi_{\mathbf{x},\mathbf{x}}$. Combining the population bounds with the uniform consistency of $\widehat{\mathbf{\Upsilon}}_{t,\mathbf{x},\mathbf{x}}$, the bound for $\widehat{\Xi} - \Xi$, and Lemma SA-3 gives $nh^d \widehat{\Xi}_{t,\mathbf{x},\mathbf{x}} \gtrsim_{\mathbb{P}} 1$ and $nh^d \widehat{\Xi}_{t,\mathbf{x},\mathbf{x}} \lesssim_{\mathbb{P}} 1$ uniformly in $\mathbf{x}$. The corresponding correlation bound follows from these lower bounds and the displayed uniform consistency of $\widehat{\Xi}_{t,\mathbf{x}_1,\mathbf{x}_2}$. □

SA-8.6 Proof of Theorem SA-1

Since $\vartheta(\mathbf{x}) = \theta_{1,\mathbf{x}}(0) - \theta_{0,\mathbf{x}}(0)$ and $\tau(\mathbf{x}) = \mu_1(\mathbf{x}) - \mu_0(\mathbf{x})$, it is enough to prove the result for one treatment assignment group $t \in \{0, 1\}$. By Assumption SA-1(iii) and Assumption SA-2(ii), for any $r \neq 0$, for any $\mathbf{x} \in \mathscr{B}$ and $\mathbf{y} \in S_{t,\mathbf{x}}(|r|)$, $|\mu_t(\mathbf{y}) - \mu_t(\mathbf{x})| \lesssim |r|$. Hence, for any $r \neq 0$, for any $\mathbf{x} \in \mathscr{B}$, $t \in \{0, 1\}$,

$$|\theta_{t,\mathbf{x}}(r) - \mu_t(\mathbf{x})| \leq \frac{\int_{S_{t,\mathbf{x}}(|r|)} |\mu_t(\mathbf{y}) - \mu_t(\mathbf{x})| f_X(\mathbf{y}) \mathfrak{H}^{d-1}(\mathrm{d}\mathbf{y})}{\int_{S_{t,\mathbf{x}}(|r|)} f_X(\mathbf{y}) \mathfrak{H}^{d-1}(\mathrm{d}\mathbf{y})} \lesssim |r|.$$

implying

$$|\theta_{t,\mathbf{x}}(0) - \mu_t(\mathbf{x})| \leq \lim_{r \to 0,\, r \in \mathscr{I}_t} |\theta_{t,\mathbf{x}}(r) - \mu_t(\mathbf{x})| = 0,$$

which establishes the result. □

SA-8.7 Proof of Theorem SA-2

By Theorem SA-1 and Equation (SA-14), we have

$$\begin{aligned}
\sup_{\mathbf{x} \in \mathscr{B}} |\mathfrak{B}_t(\mathbf{x})| &= \sup_{\mathbf{x} \in \mathscr{B}} \left| \mathbf{e}_0^\top \mathbf{\Psi}_{t,\mathbf{x}}^{-1} \mathbf{S}_{t,\mathbf{x}} - \mu_t(\mathbf{x}) \right| \\
&= \sup_{\mathbf{x} \in \mathscr{B}} \left| \mathbf{e}_0^\top \mathbf{\Psi}_{t,\mathbf{x}}^{-1} \mathbb{E}\Big[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big) K_h(D_i(\mathbf{x})) \big(\mu_t(\mathbf{X}_i) - \mu_t(\mathbf{x})\big) \mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t) \Big] \right| \\
&\lesssim \sup_{\mathbf{x} \in \mathscr{B}} \sup_{\mathbf{z} \in \mathscr{X}} |\mu_t(\mathbf{x}) - \mu_t(\mathbf{z})| \mathbb{1}(K_h(D_{\mathbf{z}}(\mathbf{x})) > 0) \\
&\lesssim h.
\end{aligned}$$

Taking the sum over $t = 0, 1$ gives $\sup_{\mathbf{x} \in \mathscr{B}} |\mathfrak{B}(\mathbf{x})| \lesssim h$. This result holds for all $d \geq 1$.

The same argument, uniformly over $\mathscr{P}$, gives the upper bound in the case of $d = 2$. Thus, it remains to establish the lower bound in the sharpness statement for $d = 2$. Let $R_K < \infty$ satisfy $\mathrm{supp}(K) \subseteq [-R_K, R_K]$ and set $\bar{R}_K = \max\{1, R_K\}$. Suppose $\mathbf{X}_i \sim \mathsf{Uniform}([-2\bar{R}_K, 2\bar{R}_K]^2)$, $\mu_0(x_1, x_2) = 0$, and $\mu_1(x_1, x_2) = x_2$ on $\mathscr{X} = [-2\bar{R}_K, 2\bar{R}_K]^2$. Let

$$\mathscr{A}_1 = \{(x, y) \in \mathscr{X} : x \geq 0,\ y \geq 0\}, \qquad \mathscr{A}_0 = \mathscr{X} \setminus \mathscr{A}_1,$$

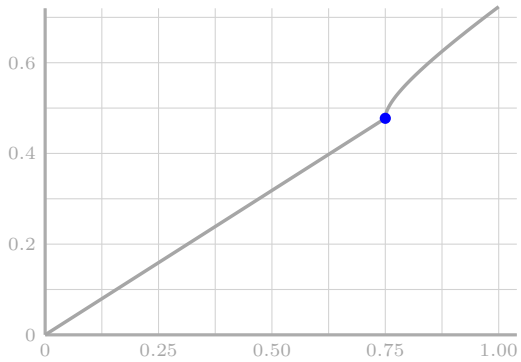


Figure SA-1: Conditional mean given distance with one kink.

and $\mathscr{B} = ([0, 2\bar{R}_K] \times \{0\}) \cup (\{0\} \times [0, 2\bar{R}_K])$. Let $\mathscr{d}$ be Euclidean distance, so $D_i(\mathbf{x}) = \|\mathbf{X}_i - \mathbf{x}\|$ on $\mathscr{A}_1$ and $D_i(\mathbf{x}) = -\|\mathbf{X}_i - \mathbf{x}\|$ on $\mathscr{A}_0$. This data-generating process belongs to the class $\mathscr{P}$ for a sufficiently large constant L. For $\mathbf{x}_s = (s, 0)$, $0 \le s \le 1$, the induced conditional mean on the treated side is

$$\theta_{1,\mathbf{x}_s}(r) = \begin{cases} 2r/\pi, & 0 \le r \le s, \\ (r+s)/(\pi - \arccos(s/r)), & r > s. \end{cases}$$

Thus the conditional mean given signed distance has a kink at $r = s$ when $s > 0$. Figure SA-1 plots this function for $s = 3/4$.

Let $\mathrm{bias}_{n,1}(h, s)$ denote the level projection bias at $\mathbf{x}_s$ with bandwidth h. A change of variables gives the scaling identity

$$\mathrm{bias}_{n,1}(Ch, Cs) = C\,\mathrm{bias}_{n,1}(h, s), \qquad 0 < C < 1, \tag{SA-18}$$

because the Gram matrix is invariant under the rescaling $(u, v, s, h) \mapsto (Cu, Cv, Cs, Ch)$, while the linear conditional mean $\mu_1(u, v) = v$ scales by C. Since $\mu_0 \equiv 0$,

$$\mathrm{bias}_{n,0}(h, s) = 0, \qquad 0 < h < 1, \quad 0 < s < h.$$

Therefore it is enough to show $\sup_{0 \le s \le 1} |\,\mathrm{bias}_{n,1}(1, s)| > 0$.

Suppressing the common density factor, write

$$\mathrm{bias}_{n,1}(1, s) = \mathbf{e}_0^\top \mathbf{A}(s)^{-1} \mathbf{B}(s),$$

where

$$\mathbf{A}(s) = \int_0^\infty \int_{\Theta_s(u)}^\pi \mathbf{r}_p(u)\mathbf{r}_p(u)^\top K(u) u \, \mathrm{d}\theta \, \mathrm{d}u,$$

$$\mathbf{B}(s) = \int_0^\infty \int_{\Theta_s(u)}^\pi \mathbf{r}_p(u) K(u) u^2 \sin(\theta) \, \mathrm{d}\theta \, \mathrm{d}u, \qquad \Theta_s(u) = \begin{cases} 0, & 0 \le u \le s, \\ \arccos(s/u), & u > s. \end{cases}$$

At $s = 0$,

$$\mathbf{A}(0) = \frac{\pi}{2}\int_0^\infty u\mathbf{r}_p(u)\mathbf{r}_p(u)^\top K(u)\,\mathrm{d}u, \qquad \mathbf{B}(0) = \int_0^\infty u^2\mathbf{r}_p(u)K(u)\,\mathrm{d}u, \tag{SA-19}$$

If $p = 0$, then $\mathbf{A}(0)$ and $\mathbf{B}(0)$ are scalars and $\mathbf{e}_0^\top\mathbf{A}(0)^{-1}\mathbf{B}(0) > 0$, so $\sup_{0\le s\le 1}|\operatorname{bias}_{n,1}(1,s)| > 0$ immediately. It remains to consider $p \ge 1$. Differentiating under the integral sign gives

$$\dot{\mathbf{A}}(0) = \int_0^\infty \mathbf{r}_p(u)\mathbf{r}_p(u)^\top K(u)\,\mathrm{d}u, \qquad \dot{\mathbf{B}}(0) = \int_0^\infty u\mathbf{r}_p(u)K(u)\,\mathrm{d}u.$$

Using $\mathbf{A}(0)^{-1}\mathbf{B}(0) = (2/\pi)\mathbf{e}_1$, where $\mathbf{e}_1$ is the slope selector in the polynomial basis, and $\mathbf{A}(0)^{-1}\dot{\mathbf{B}}(0) = (2/\pi)\mathbf{e}_0$,

$$\begin{aligned}\frac{d}{ds}\operatorname{bias}_{n,1}(1,s)\Big|_{s=0} &= -\mathbf{e}_0^\top\mathbf{A}(0)^{-1}\dot{\mathbf{A}}(0)\mathbf{A}(0)^{-1}\mathbf{B}(0) + \mathbf{e}_0^\top\mathbf{A}(0)^{-1}\dot{\mathbf{B}}(0)\\ &= -\frac{4}{\pi^2} + \frac{2}{\pi} \neq 0.\end{aligned} \tag{SA-20}$$

By continuity, the same strict inequality holds for $p \ge 1$. Thus, for all $p \ge 0$, $\sup_{0\le s\le 1}|\operatorname{bias}_{n,1}(1,s)| > 0$. Combining this with (SA-18) yields

$$\inf_{0<h<1}\sup_{\mathbf{x}\in\mathscr{B}}\frac{|\mathfrak{B}_1(\mathbf{x}) - \mathfrak{B}_0(\mathbf{x})|}{h} \ge \inf_{0<h<1}\sup_{0<s<h}\left|\operatorname{bias}_{n,1}\left(1,\frac{s}{h}\right)\right| > 0.$$

This proves the lower bound and completes the proof. □

SA-8.8 Proof of Theorem SA-3

Part (i): High-level smoothness condition.

It is enough to prove the result for one side t. Since K is compactly supported, take $R_K < \infty$ such that $\operatorname{supp}(K) \subseteq [-R_K, R_K]$; it suffices to consider $|r| \le R_K h$ with $r \in \mathscr{I}_t$. Taylor's theorem gives

$$\theta_{t,\mathbf{x}}(r) = \sum_{j=0}^{m-1}\frac{\theta_{t,\mathbf{x}}^{(j)}(0)}{j!}r^j + R_{t,\mathbf{x}}(r), \qquad \sup_{\mathbf{x}\in\mathscr{B}}|R_{t,\mathbf{x}}(r)| \lesssim |r|^m.$$

The local polynomial fit of order p reproduces every polynomial of degree at most p. Since $m \le p+1$, the level bias is therefore generated only by the remainder. Using $\|\mathbf{\Psi}_{t,\mathbf{x}}^{-1}\| \lesssim 1$ and the bounded kernel moments,

$$\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}_t(\mathbf{x})| \lesssim \sup_{\mathbf{x}\in\mathscr{B}}\mathbb{E}\Big[K_h(D_i(\mathbf{x}))|D_i(\mathbf{x})|^m\mathbb{1}(D_i(\mathbf{x})\in\mathscr{I}_t)\Big] \lesssim h^m.$$

Taking the sum over $t = 0, 1$ gives the uniform result. The pointwise result is identical with the outer supremum removed.

Part (ii): Pointwise bivariate smooth-boundary primitive condition.

Fix $\mathbf{x} \in \mathscr{B}$ and let $m_{\mathbf{x}} = \kappa \wedge (p+1)$. It is enough to verify the primitive smoothness condition in part (i) for one treatment side t at this fixed boundary point; the treatment-effect bias is the difference of the two side-specific biases.

After re-centering and rotating coordinates, set $\mathbf{x} = 0$ and parameterize the local curve by arc length so that $\gamma(0) = 0$ and $\gamma'(0) = (1, 0)$. For u near zero,

$$\phi(u) = \|\gamma(u) - \gamma(0)\|^2 = u^2 a(u), \qquad a(0) = 1,$$

where a is C^κ and bounded away from zero in a neighborhood of zero. Therefore the two local solutions of $\phi(u) = r^2$ can be written as $u_+(r)$ and $u_-(r)$, where $u_+(r) = rq_+(r)$ and $u_-(r) = -rq_-(r)$, with q_+ and q_- being C^κ functions. This follows from the implicit function theorem applied to $u/\sqrt{\phi(u)}$ on the positive and negative sides of zero.

The polar angles at which the circle $\{\mathbf{u} : \|\mathbf{u} - \mathbf{x}\| = r\}$ intersects the local boundary are

$$\alpha_+(\mathbf{x}, r) = \arg\{\gamma(u_+(r)) - \mathbf{x}\}, \qquad \alpha_-(\mathbf{x}, r) = \arg\{\gamma(u_-(r)) - \mathbf{x}\}.$$

The preceding representation and the smoothness of γ imply that $\alpha_+(\mathbf{x}, \cdot)$ and $\alpha_-(\mathbf{x}, \cdot)$ are C^κ on $[0, \varepsilon]$, after possibly shrinking ε. The local angular interval corresponding to $\mathscr{A}_t$ therefore has endpoints $\alpha_{t,1}(\mathbf{x}, r)$ and $\alpha_{t,2}(\mathbf{x}, r)$ that are C^κ in r.

For $r > 0$,

$$\theta_{t,\mathbf{x}}(r) = \frac{\int_{\alpha_{t,1}(\mathbf{x},r)}^{\alpha_{t,2}(\mathbf{x},r)} \mu_t(\mathbf{x} + r(\cos\theta, \sin\theta)) f_X(\mathbf{x} + r(\cos\theta, \sin\theta))\, \mathrm{d}\theta}{\int_{\alpha_{t,1}(\mathbf{x},r)}^{\alpha_{t,2}(\mathbf{x},r)} f_X(\mathbf{x} + r(\cos\theta, \sin\theta))\, \mathrm{d}\theta}.$$

The numerator and denominator are $C^{m_\mathbf{x}}$ in r by repeated application of Leibniz's rule, because μ_t, f_X, and the angle endpoints have the required smoothness. The denominator is bounded away from zero for small r because the angular interval length converges to π and f_X is bounded away from zero. Hence $\theta_{t,\mathbf{x}}(\cdot)$ is $m_\mathbf{x}$-times continuously differentiable near zero. Applying the pointwise version of part (i) gives

$$|\mathfrak{B}(\mathbf{x})| \lesssim h^{m_\mathbf{x}}.$$

Part (iii): Uniform bivariate smooth-boundary primitive condition.

Let $m = \iota \wedge (p + 1)$. We verify the primitive smoothness condition in part (i) uniformly over $\mathbf{x} \in \mathscr{B}^o$.

Fix $\mathbf{x}_0 = \gamma(s_0) \in \mathscr{B}^o$ and re-center and rotate coordinates so that $\mathbf{x}_0 = 0$ and $\gamma'(s_0) = (1, 0)$ after reparameterizing the curve by arc length. Since γ is $C^{\iota+2}$, for u near zero,

$$\phi(u) = \|\gamma(s_0 + u) - \gamma(s_0)\|^2 = u^2 a(u), \qquad a(0) = 1,$$

where a is C^ι and bounded away from zero uniformly for $s_0 \in [\delta, L - \delta]$. Therefore the two local solutions of $\phi(u) = r^2$ can be written as $u_+(r)$ and $u_-(r)$, where $u_+(r) = rq_+(r)$ and $u_-(r) = -rq_-(r)$ with q_+ and q_- being C^ι functions whose derivatives up to order ι are uniformly bounded. This follows from the implicit function theorem applied to $u/\sqrt{\phi(u)}$ on the positive and negative sides of zero.

The polar angles at which the circle $\{\mathbf{u} : \|\mathbf{u} - \mathbf{x}_0\| = r\}$ intersects the boundary are therefore

$$\alpha_+(\mathbf{x}_0, r) = \arg\{\gamma(s_0 + u_+(r)) - \mathbf{x}_0\}, \qquad \alpha_-(\mathbf{x}_0, r) = \arg\{\gamma(s_0 + u_-(r)) - \mathbf{x}_0\}.$$

The preceding representation and the smoothness of γ imply that $\alpha_+(\mathbf{x}_0, \cdot)$ and $\alpha_-(\mathbf{x}_0, \cdot)$ are C^ι on $[0, \varepsilon]$, with derivatives uniformly bounded over $\mathbf{x}_0 \in \mathscr{B}^o$. By the assumed two-intersection condition, the angular interval

corresponding to $\mathscr{A}_t$ has endpoints given by these two angle functions, up to a fixed ordering convention.

For $r > 0$, write this interval as $[\alpha_{t,1}(\mathbf{x}_0, r), \alpha_{t,2}(\mathbf{x}_0, r)]$. Then

$$\theta_{t,\mathbf{x}_0}(r) = \frac{\int_{\alpha_{t,1}(\mathbf{x}_0,r)}^{\alpha_{t,2}(\mathbf{x}_0,r)} \mu_t(\mathbf{x}_0 + r(\cos\theta, \sin\theta)) f_X(\mathbf{x}_0 + r(\cos\theta, \sin\theta))\, \mathrm{d}\theta}{\int_{\alpha_{t,1}(\mathbf{x}_0,r)}^{\alpha_{t,2}(\mathbf{x}_0,r)} f_X(\mathbf{x}_0 + r(\cos\theta, \sin\theta))\, \mathrm{d}\theta}.$$

The numerator and denominator are C^m in r by repeated application of Leibniz's rule, because μ_t, f_X, and the angle endpoints have uniformly bounded derivatives up to order m. The denominator is bounded away from zero uniformly for small r because the angular interval length converges uniformly to π and f_X is bounded away from zero; Assumption SA-2(v) ensures that the corresponding Hausdorff-measure ratio is well defined before this local parametrization. Hence $\theta_{t,\mathbf{x}_0}$ is C^m on $[0, \varepsilon]$, and its derivatives up to order m are uniformly bounded over $\mathbf{x}_0 \in \mathscr{B}^o$.

Applying the argument in part (i) on the index set $\mathscr{B}^o$ gives

$$\sup_{\mathbf{x}\in\mathscr{B}^o} |\mathfrak{B}(\mathbf{x})| \lesssim h^m.$$

□

SA-8.9 Proof of Theorem SA-4

For a fixed $\mathbf{x} \in \mathscr{B}$, the proofs of Lemmas SA-3 and SA-4 can be done when the index set is the singleton $\{\mathbf{x}\}$ instead of $\mathscr{B}$, replacing Chernozhukov et al. [2014b, Corollary 5.1] with Bernstein's inequality and obtaining

$$\left|\mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}}^{-1} \mathbf{O}_{t,\mathbf{x}}\right| \lesssim_{\mathbb{P}} \sqrt{\frac{1}{nh^d}} + \frac{1}{n^{\frac{1+v}{2+v}} h^d},$$
$$\left|\mathbf{e}_0^\top (\widehat{\boldsymbol{\Psi}}_{t,\mathbf{x}}^{-1} - \boldsymbol{\Psi}_{t,\mathbf{x}}^{-1}) \mathbf{O}_{t,\mathbf{x}}\right| \lesssim_{\mathbb{P}} \sqrt{\frac{1}{nh^d}} \left(\sqrt{\frac{1}{nh^d}} + \frac{1}{n^{\frac{1+v}{2+v}} h^d}\right).$$

The pointwise convergence rate follows from decomposition (SA-1). The uniform statement follows from Lemmas SA-3 and SA-4 and the same decomposition. □

SA-8.10 Proof of Theorem SA-5

Define $\overline{\mathrm{T}}(\mathbf{x}) = \sum_{i=1}^n Z_i$, with $Z_i = Z_{1,i} - Z_{0,i}$ independent random variables ($i = 1, 2, \ldots, n$),

$$Z_{t,i} = \frac{1}{n} \big(\Xi_{\mathbf{x},\mathbf{x}}\big)^{-1/2} \mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}}^{-1} \mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big) K_h(D_i(\mathbf{x}))(Y_i - \theta_{t,\mathbf{x}}^*(D_i(\mathbf{x})))\mathbb{1}(\mathbf{X}_i \in \mathscr{A}_t),$$

$\mathbb{E}[Z_i] = 0$ and $\mathbb{V}[Z_i] = n^{-1}$. By the Berry-Esseen theorem,

$$\sup_{u\in\mathbb{R}} \Big|\mathbb{P}\big(\overline{\mathrm{T}}(\mathbf{x}) \le u\big) - \Phi(u)\Big| \lesssim \sum_{i=1}^n \mathbb{E}\big[|Z_i|^3\big] \lesssim \sum_{i=1}^n \mathbb{E}\big[|Z_{1,i}|^3\big] + \sum_{i=1}^n \mathbb{E}\big[|Z_{0,i}|^3\big]$$

where

$$\sum_{i=1}^n \mathbb{E}\big[|Z_{t,i}|^3\big] = n^{-2} \big(\Xi_{\mathbf{x},\mathbf{x}}\big)^{-3/2} \mathbb{E}\Big[\Big|\mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}}^{-1} \mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big) K_h(D_i(\mathbf{x}))\mathbb{1}(\mathbf{X}_i \in \mathscr{A}_t)(Y_i - \theta_{t,\mathbf{x}}^*(D_i(\mathbf{x}))\Big|^3\Big]$$

$$
\begin{aligned}
&\lesssim n^{-2}\big(\Xi_{\mathbf{x},\mathbf{x}}\big)^{-3/2}\mathbb{E}[|K_h(D_i(\mathbf{x}))|^3|Y_i-\theta^*_{t,\mathbf{x}}(D_i(\mathbf{x}))|^3]\\
&\lesssim n^{-2}\big(\Xi_{\mathbf{x},\mathbf{x}}\big)^{-3/2}\mathbb{E}[|K_h(D_i(\mathbf{x}))|^3(\mathbb{E}[|Y_i|^3|\mathbf{X}_i]+|\theta^*_{t,\mathbf{x}}(D_i(\mathbf{x}))|^3)]\\
&\lesssim (nh^d)^{-1/2},
\end{aligned}
$$

noting that $K_h(D_i(\mathbf{x}))$ is supported on a set of probability $O(h^d)$ and satisfies $|K_h(D_i(\mathbf{x}))|\lesssim h^{-d}$, $\Xi_{\mathbf{x},\mathbf{x}}\gtrsim (nh^d)^{-1}$ by the pointwise version of the covariance lower-bound argument in Lemma SA-5, $\mathbb{E}[|Y_i|^3|\mathbf{X}_i]\lesssim 1$ by Assumption SA-1(v), and $\max_{1\le i\le n}\sup_{\mathbf{x}\in\mathscr{B}}|\theta^*_{t,\mathbf{x}}(D_i(\mathbf{x}))|\lesssim 1$ because

$$
\theta^*_{t,\mathbf{x}}(D_i(\mathbf{x}))=\boldsymbol{\gamma}^*_t(\mathbf{x})^\top\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)=\big(\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{S}_{t,\mathbf{x}}\big)^\top\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big).
$$

Since

$$
\big|\widehat{\mathrm{T}}(\mathbf{x})-\overline{\mathrm{T}}(\mathbf{x})\big|\lesssim_{\mathbb{P}}\frac{1}{\sqrt{nh^d}}+\frac{1}{n^{\frac{v}{2+v}}h^d}+\sqrt{nh^d}|\mathfrak{B}(\mathbf{x})|,
$$

and the same pointwise covariance argument gives $\widehat{\Xi}_{\mathbf{x},\mathbf{x}}/\Xi_{\mathbf{x},\mathbf{x}}\to_{\mathbb{P}}1$ under the conditions imposed, pointwise asymptotic normality follows. Finally, validity of the confidence interval estimator is immediate. □

SA-8.11 Proof of Theorem SA-6

Because $v>0$, the condition $n^{\frac{v}{2+v}}h^d/\log(1/h)\to\infty$ implies $nh^d/\log(1/h)\to\infty$, so the preceding uniform approximation lemmas apply. We make the decomposition based on Equation (SA-1) and convergence of $\widehat{\Xi}_{\mathbf{x},\mathbf{x}}$,

$$
\begin{aligned}
\widehat{\mathrm{T}}(\mathbf{x})-\overline{\mathrm{T}}(\mathbf{x})&=\big(\widehat{\Xi}_{\mathbf{x},\mathbf{x}}\big)^{-1/2}\Bigg(\sum_{t\in\{0,1\}}(2t-1)(\widehat{\theta}_{t,\mathbf{x}}(0)-\theta_{t,\mathbf{x}}(0))\Bigg)\\
&\quad-\big(\Xi_{\mathbf{x},\mathbf{x}}\big)^{-1/2}\Bigg(\sum_{t\in\{0,1\}}(2t-1)\mathbf{e}_0^\top\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}\Bigg)\\
&=\big(\widehat{\Xi}_{\mathbf{x},\mathbf{x}}\big)^{-1/2}\Bigg(\sum_{t\in\{0,1\}}(2t-1)(\widehat{\theta}_{t,\mathbf{x}}(0)-\theta_{t,\mathbf{x}}(0))\\
&\quad-\sum_{t\in\{0,1\}}(2t-1)\mathbf{e}_0^\top\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}\Bigg)\qquad(=\Delta_{1,\mathbf{x}})\\
&\quad+\big((\widehat{\Xi}_{\mathbf{x},\mathbf{x}})^{-1/2}-(\Xi_{\mathbf{x},\mathbf{x}})^{-1/2}\big)\sum_{t\in\{0,1\}}(2t-1)\mathbf{e}_0^\top\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}\qquad(=\Delta_{2,\mathbf{x}})
\end{aligned}
$$

By Lemma SA-3 and SA-4, and the decomposition Equation (SA-1),

$$
\begin{aligned}
&\sup_{\mathbf{x}\in\mathscr{B}}\Bigg|\sum_{t\in\{0,1\}}(2t-1)(\widehat{\theta}_{t,\mathbf{x}}(0)-\theta_{t,\mathbf{x}}(0))-\sum_{t\in\{0,1\}}(2t-1)\mathbf{e}_0^\top\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{O}_{t,\mathbf{x}}\Bigg|\\
&\lesssim_{\mathbb{P}}\sqrt{\frac{\log(1/h)}{nh^d}}\left(\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d}\right)+\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}(\mathbf{x})|.
\end{aligned}
$$

Together with Lemma SA-5,

$$\sup_{\mathbf{x}\in\mathscr{B}}|\Delta_{1,\mathbf{x}}| \lesssim_{\mathbb{P}} \frac{\log(1/h)}{\sqrt{nh^d}} + \frac{(\log(1/h))^{\frac{3}{2}}}{n^{\frac{1+v}{2+v}}h^d} + \sqrt{nh^d}\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}(\mathbf{x})|. \tag{SA-21}$$

By Lemma SA-3, Lemma SA-4 and Lemma SA-5, and assume $\frac{n^{\frac{v}{2+v}}h^d}{\log(1/h)}\to\infty$, then

$$\begin{aligned}
\sup_{\mathbf{x}\in\mathscr{B}}\left|\mathbf{e}_0^\top \boldsymbol{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{O}_{t,\mathbf{x}}\left((\Xi_{\mathbf{x},\mathbf{x}})^{-1/2}-(\widehat{\Xi}_{\mathbf{x},\mathbf{x}})^{-1/2}\right)\right| &\lesssim_{\mathbb{P}} \sqrt{nh^d}\left(\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{1+v}{2+v}}h^d}\right)\left(\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}\right)\\
&= \sqrt{\log(1/h)}\left(1+\sqrt{\frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}}\right)\left(\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}\right)\\
&\lesssim \sqrt{\log(1/h)}\left(\sqrt{\frac{\log(1/h)}{nh^d}}+\frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}\right).
\end{aligned}$$

Hence

$$\sup_{\mathbf{x}\in\mathscr{B}}|\Delta_{2,\mathbf{x}}| \lesssim_{\mathbb{P}} \frac{\log(1/h)}{\sqrt{nh^d}} + \frac{(\log(1/h))^{\frac{3}{2}}}{n^{\frac{v}{2+v}}h^d}. \tag{SA-22}$$

Putting together Equations (SA-21), (SA-22) give the result. □

SA-8.12 Proof of Theorem SA-7

We will verify the high level conditions stated in Theorem SA-25.

By the rectangular-support condition, after a componentwise affine rescaling we can take $\mathcal{X}=[0,1]^d$. Since f_X is continuous and bounded away from zero on this compact rectangle, Cattaneo and Yu [2025, Section 3.1] implies that $\mathcal{Q}_{\mathscr{G}\cup\mathscr{M}}=\mathbb{P}_X$ is a valid surrogate measure for $\mathbb{P}_X$ with respect to $\mathscr{G}\cup\mathscr{M}$, and that the Rosenblatt transform $\phi_{\mathscr{G}\cup\mathscr{M}}=T_{\mathbb{P}_X}$ is a valid normalizing transformation. The constants $\mathtt{c}_1$ and $\mathtt{c}_2$ from Theorem SA-25 are therefore finite and uniformly bounded by constants depending only on the density bounds and the support geometry, and are absorbed in the displays below.

Recall $\mathscr{G}=\{g_{\mathbf{x}}:\mathbf{x}\in\mathscr{B}\}$ where

$$g_{\mathbf{x}}(\mathbf{u}) = \mathbb{1}(\mathbf{u}\in\mathscr{A}_1)\mathfrak{K}_1(\mathbf{u};\mathbf{x}) - \mathbb{1}(\mathbf{u}\in\mathscr{A}_0)\mathfrak{K}_0(\mathbf{u};\mathbf{x}).$$

By the same signed-distance entropy and localization arguments as in Lemma SA-3, combined with [Cattaneo et al., 2024, Lemma 7] in the Lipschitz-kernel case and the VC-ball argument in the uniform-kernel case, we get properties of $\mathscr{G}$ as follows:

$$\mathtt{M}_{\mathscr{G}}\lesssim h^{-d/2},\qquad \mathtt{E}_{\mathscr{G}}\lesssim h^{d/2},\qquad \mathtt{TV}_{\mathscr{G}}\lesssim h^{d/2-1},\qquad \sup_Q N(\mathscr{G},\|\cdot\|_{Q,2},\varepsilon(2c+1)^{d+1}\mathtt{M}_{\mathscr{G}})\le 2\mathbf{c}'\varepsilon^{-d-1}+2.$$

By definition of $\theta^*_{t,\mathbf{x}}(\cdot)$, for each $\mathbf{x}\in\mathscr{B}$, $t\in\{0,1\}$,

$$\theta^*_{t,\mathbf{x}}(D_{\mathbf{u}}(\mathbf{x})) = \boldsymbol{\gamma}^*_t(\mathbf{x})^\top\mathbf{r}_p\Big(\frac{D_{\mathbf{u}}(\mathbf{x})}{h}\Big) = (\boldsymbol{\Psi}^{-1}_{t,\mathbf{x}}\mathbf{S}_{t,\mathbf{x}})^\top\mathbf{r}_p\Big(\frac{D_{\mathbf{u}}(\mathbf{x})}{h}\Big),$$

recalling

$$\boldsymbol{\Psi}_{t,\mathbf{x}} = \mathbb{E}\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)^\top K_h(D_i(\mathbf{x}))\mathbb{1}(D_i(\mathbf{x}) \in \mathscr{I}_t)\right], \quad \mathbf{S}_{t,\mathbf{x}} = \mathbb{E}\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)K_h(D_i(\mathbf{x}))Y_i\mathbb{1}(\mathbf{X}_i \in \mathscr{A}_t)\right].$$

The preceding bounds imply that $\left\|\boldsymbol{\Psi}_{t,\mathbf{x}}^{-1}\right\| \lesssim 1$, $\|\mathbf{S}_{t,\mathbf{x}}\| \lesssim 1$, and

$$\mathtt{M}_{\mathscr{M}_t} \lesssim h^{-d/2}, \qquad \mathtt{E}_{\mathscr{M}_t} \lesssim h^{d/2}, \qquad t \in \{0,1\}.$$

In what follows, we verify the entropy and total variation properties of $\mathscr{M}$. Using the product rule, we have

$$\sup_{\mathbf{u}\in\mathscr{X}}\sup_{\mathbf{x},\mathbf{x}'\in\mathscr{B}}\frac{|\theta^*_{t,\mathbf{x}}(D_\mathbf{u}(\mathbf{x})) - \theta^*_{t,\mathbf{x}}(D_\mathbf{u}(\mathbf{x}'))|}{\|\mathbf{x}-\mathbf{x}'\|} \lesssim h^{-1}.$$

Define $f_{t,\mathbf{x}}(\cdot) = \frac{h^{-d/2}}{\sqrt{n\Xi_{\mathbf{x},\mathbf{x}}}}\mathbf{e}_0^\top\boldsymbol{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{r}_p(\cdot)\,K(\cdot)(\boldsymbol{\Psi}_{t,\mathbf{x}}^{-1}\mathbf{S}_{t,\mathbf{x}})^\top\mathbf{r}_p(\cdot)$. Then,

$$\mathfrak{K}_t(\mathbf{u};\mathbf{x})\theta^*_{t,\mathbf{x}}(D_\mathbf{u}(\mathbf{x})) = h^{-d/2}f_{t,\mathbf{x}}\Big(\frac{D_\mathbf{u}(\mathbf{x})}{h}\Big), \qquad \mathbf{u}\in\mathscr{X}, \mathbf{x}\in\mathscr{B}, t\in\{0,1\}.$$

Take $\mathscr{M}_t = \{\mathfrak{K}_t(\cdot;\mathbf{x})\theta^*_{t,\mathbf{x}}(D_\cdot(\mathbf{x})) : \mathbf{x}\in\mathscr{B}\}$, $t\in\{0,1\}$. If K is Lipschitz, then for $t\in\{0,1\}$, $f_{t,\mathbf{x}}$ satisfies:

(i) boundedness	$\sup_{\mathbf{x}\in\mathscr{B}}\sup_{u\in\mathbb{R}}\lvert f_{t,\mathbf{x}}(u)\rvert \le \mathbf{c}$,
(ii) compact support	$\operatorname{supp}(f_{t,\mathbf{x}}(\cdot)) \subseteq [-\mathbf{c},\mathbf{c}], \forall\mathbf{x}\in\mathscr{B}$,
(iii) Lipschitz continuity	$\sup_{\mathbf{x}\in\mathscr{B}}\sup_{u,u'\in\mathbb{R}}\frac{\lvert f_{t,\mathbf{x}}(u)-f_{t,\mathbf{x}}(u')\rvert}{\lvert u-u'\rvert} \le \mathbf{c}$,
	$\sup_{u\in\mathbb{R}}\sup_{\mathbf{x},\mathbf{x}'\in\mathscr{B}}\frac{\lvert f_{t,\mathbf{x}}(u)-f_{t,\mathbf{x}'}(u)\rvert}{\|\mathbf{x}-\mathbf{x}'\|} \le \mathbf{c}h^{-1}$,

for some constant $\mathbf{c}$ not depending on n. Then, by an argument similar to Cattaneo et al. [2024, Lemma 7], there exists a constant $\mathbf{c}'$ depending only on $\mathbf{c}$ and d such that, for any $0\le\varepsilon\le 1$,

$$\sup_Q N\left(h^{d/2}\mathscr{M}_t, \|\cdot\|_{Q,1}, (2\mathbf{c}+1)^{d+1}\varepsilon\right) \le \mathbf{c}'\varepsilon^{-d-1}+1,$$

where the supremum is taken over all finite discrete measures. If $K(u) = \mathbb{1}(u\in[-1,1])$, the same display follows by replacing the preceding Lipschitz step with the VC-ball argument used in Lemma SA-3; the polynomial factors and $\theta^*_{t,\mathbf{x}}(D_\cdot(\mathbf{x}))$ are uniformly bounded on the support of $K_h(D_\cdot(\mathbf{x}))$, and products with the fixed side indicator preserve VC-type entropy. Taking a constant envelope function $\mathtt{M}_{\mathscr{M}_t} = (2\mathbf{c}+1)^{d+1}h^{-d/2}$, we have for any $0<\varepsilon\le 1$,

$$\sup_Q N\left(\mathscr{M}_t, \|\cdot\|_{Q,1}, \varepsilon\mathtt{M}_{\mathscr{M}_t}\right) \le \mathbf{c}'\varepsilon^{-d-1}+1.$$

By Lemma SA-11, above implies the uniform covering number for $\mathscr{M}_t$ satisfies

$$\mathtt{N}_{\mathscr{M}_t}(\varepsilon) \le 4\mathbf{c}'(\varepsilon/2)^{-d-1}, \qquad 0<\varepsilon\le 1.$$

Since $\mathscr{M}\subseteq\mathscr{M}_0+\mathscr{M}_1$, here $+$ denotes the Minkowski sum, with $\mathtt{M}_{\mathscr{M}}$ taken to be $\mathtt{M}_{\mathscr{M}_0}+\mathtt{M}_{\mathscr{M}_1}$, a bound on the

uniform covering number of $\mathscr{M}$ can be given by

$$\mathtt{N}_{\mathscr{M}}(\varepsilon) \leq 16(\mathbf{c}')^2(\varepsilon/2)^{-2d-2}, \qquad 0 < \varepsilon \leq 1.$$

With the assumption that $\mathscr{L}(E_{t,\mathbf{x}}(h)) \leq Ch^{d-1}$ for $E_{t,\mathbf{x}}(h) = \{\mathbf{y} \in \mathscr{A}_t : \varsigma_t \mathscr{d}(\mathbf{y},\mathbf{x})/h \in \mathrm{supp}(K)\}$ for all $t \in \{0,1\}$, $\mathbf{x} \in \mathscr{B}$, and the fact that $\mathtt{TV}_{\mathscr{M}_t} \lesssim h^{d/2-1}$ for $t \in \{0,1\}$, the same argument as in the paragraph **Total Variation** in the proof of Theorem SA-25 shows

$$\mathtt{TV}_{\mathscr{M}} \lesssim h^{d/2-1}.$$

Now apply Theorem SA-25 with $\mathscr{G}$, $\mathscr{M}$ defined in Equation (SA-2), $\mathscr{R} = \{\mathrm{Id}\}$, $\mathscr{S} = \{1\}$, noting that

$$\overline{\mathrm{T}}(\mathbf{x}) = A_n(g_{\mathbf{x}}, m_{\mathbf{x}}, \mathrm{Id}, 1), \qquad (g_{\mathbf{x}}, m_{\mathbf{x}}) \in \mathscr{G} \times \mathscr{M}, \quad \mathbf{x} \in \mathscr{B},$$

the result follows. □

Lemma SA-11 (VC Class to VC2 Class)**.** *Assume $\mathscr{F}$ is a VC class on a measure space $(\mathscr{X}, \mathscr{B})$: there exists an envelope function F and positive constants $c(\mathscr{F}), d(\mathscr{F})$ such that for all $\varepsilon \in (0,1)$,*

$$\sup_Q N(\mathscr{F}, \|\cdot\|_{Q,1}, \varepsilon \|F\|_{Q,1}) \leq c(\mathscr{F})\varepsilon^{-d(\mathscr{F})},$$

where the supremum is taken over all finite discrete measures. Then, $\mathscr{F}$ is also VC2 class: for all $\varepsilon \in (0,1)$,

$$\sup_Q N(\mathscr{F}, \|\cdot\|_{Q,2}, \varepsilon \|F\|_{Q,2}) \leq c(\mathscr{F})(\varepsilon^2/2)^{-d(\mathscr{F})},$$

where the supremum is taken over all finite discrete measures.

Proof of Lemma SA-11. Let Q be a finite discrete probability measure. Let $f, g \in \mathscr{F}$. Then, $\int |f-g|^2 \,\mathrm{d}Q \leq 2\int |f-g||F| \,\mathrm{d}Q$. Define another probability measure $\tilde{Q}(c_k) = F(c_k)Q(c_k)/\|F\|_{Q,1}$ on the support of Q, denoted by $\{c_1, \ldots, c_k, \ldots\}$. Then,

$$\int |f-g|^2 \,\mathrm{d}Q \leq 2\|F\|_{Q,1} \int |f-g| \,\mathrm{d}\tilde{Q} \leq 2\|F\|_{Q,1} \|f-g\|_{\tilde{Q},1}.$$

Hence, if we take an $\varepsilon^2/2$-net in $(\mathscr{F}, \|\cdot\|_{\tilde{Q},1})$ with cardinality no greater than $c(\mathscr{F})(\varepsilon^2/2)^{-d(\mathscr{F})}$, then for any $f \in \mathscr{F}$, there exists a $g \in \mathscr{F}$ such that $\|f-g\|_{\tilde{Q},1} \leq \varepsilon^2/2\|F\|_{\tilde{Q},1}$, and hence

$$\|f-g\|_{Q,2}^2 \leq 2\varepsilon^2/2\|F\|_{Q,1}\|F\|_{\tilde{Q},1} \leq \varepsilon^2\|F\|_{Q,2}^2,$$

which gives the result. □

SA-8.13 Proof of Theorem SA-8

The proof is divided into three theorem-specific technical lemmas. We first record a generic comparison lemma that translates uniform process approximations into Kolmogorov-distance bounds for suprema.

Lemma SA-12 (Generic Suprema Comparison)**.** *Let $\mathbb{T}_j = (\mathbb{T}_j(\mathbf{x}) : \mathbf{x} \in \mathscr{B})$, $j = 1, 2$, be real-valued processes*

defined on a common probability space. Write

$$M_j = \sup_{\mathbf{x}\in\mathscr{B}} |\mathbb{T}_j(\mathbf{x})|, \quad j = 1, 2, \qquad \Delta_{12} = \sup_{\mathbf{x}\in\mathscr{B}} |\mathbb{T}_1(\mathbf{x}) - \mathbb{T}_2(\mathbf{x})|.$$

For any $\eta > 0$,

$$\sup_{u\in\mathbb{R}} |\mathbb{P}(M_1 \le u) - \mathbb{P}(M_2 \le u)| \le \mathsf{AC}_2(\eta) + \mathbb{P}(\Delta_{12} > \eta),$$

where

$$\mathsf{AC}_2(\eta) = \sup_{u\in\mathbb{R}} \mathbb{P}(|M_2 - u| \le \eta).$$

If, in addition, $\mathbb{T}_2$ *is a separable mean-zero Gaussian process and the Gaussian anti-concentration bound applies to* M_2*, then*

$$\sup_{u\in\mathbb{R}} |\mathbb{P}(M_1 \le u) - \mathbb{P}(M_2 \le u)| \le 4\eta\{\mathbb{E}[M_2] + 1\} + \mathbb{P}(\Delta_{12} > \eta).$$

Consequently, if $\mathbb{E}[\Delta_{12}] \le r_n$*, then the last probability can be bounded by* r_n/η.

Proof of Lemma SA-12. On the event $\{\Delta_{12} \le \eta\}$, the inequalities

$$M_1 \le u \implies M_2 \le u + \eta, \qquad M_2 \le u - \eta \implies M_1 \le u$$

hold for every $u \in \mathbb{R}$. Hence

$$\begin{aligned}
\mathbb{P}(M_1 \le u) - \mathbb{P}(M_2 \le u) &\le \mathbb{P}(u < M_2 \le u + \eta) + \mathbb{P}(\Delta_{12} > \eta),\\
\mathbb{P}(M_2 \le u) - \mathbb{P}(M_1 \le u) &\le \mathbb{P}(u - \eta < M_2 \le u) + \mathbb{P}(\Delta_{12} > \eta).
\end{aligned}$$

Taking absolute values and then the supremum over u gives the first claim. The second display follows from the Gaussian anti-concentration inequality of Chernozhukov et al. [2014a, Theorem 2.1], applied to the symmetrized Gaussian process whose supremum is M_2. The final claim follows from Markov's inequality. □

For the theorem-specific lemmas below, define

$$\mathfrak{R}_{\mathtt{sa}} = (\log n)^{\frac{3}{2}} \Big(\frac{1}{nh^d}\Big)^{\frac{1}{2d+2}\cdot\frac{v}{v+2}} + \log(n)\Big(\frac{1}{n^{\frac{v}{2+v}}h^d}\Big)^{1/2},$$

the stochastic-linearization remainder

$$\mathfrak{R}_{\mathtt{lin}} = \sqrt{\log(1/h)}\left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}\right) + \sqrt{nh^d}\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}(\mathbf{x})|,$$

and the covariance-comparison rate

$$\mathfrak{R}_{\mathtt{cc}} = \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}}h^d}.$$

Lemma SA-13 (KS Distance Between $\overline{\mathrm{T}}$ and Z)**.** *Suppose the conditions of Theorem SA-8 hold. Then*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\overline{\mathrm{T}}(\mathbf{x})|\le u\right)-\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|\le u\right)\right|\lesssim(\log n)^{1/4}\mathfrak{R}_{\mathtt{sa}}^{1/2}.$$

Proof of Lemma SA-13. Let $a_n>0$ be chosen below. Applying Lemma SA-12 with $\mathbb{T}_1=\overline{\mathrm{T}}$ and $\mathbb{T}_2=Z$, and using Theorem SA-7, gives

$$\begin{aligned}&\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\overline{\mathrm{T}}(\mathbf{x})|\le u\right)-\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|\le u\right)\right|\\&\quad\le 4a_n\left(\mathbb{E}\left[\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|\right]+1\right)+\frac{C\mathfrak{R}_{\mathtt{sa}}}{a_n}.\end{aligned}$$

The Gaussian process Z has uniformly bounded pointwise variance. Moreover, by the same Lipschitz calculation for the distance-indexed local-polynomial weights used in Theorem SA-7, its canonical metric satisfies

$$\rho(\mathbf{x},\mathbf{y})=\mathbb{E}\left[\{Z(\mathbf{x})-Z(\mathbf{y})\}^2\right]^{1/2}\lesssim h^{-1}\|\mathbf{x}-\mathbf{y}\|_\infty.$$

Since $\mathscr{B}$ is compact and $\liminf_{n\to\infty}\log h/\log n>-\infty$,

$$\log N(\varepsilon,\mathscr{B},\rho)\lesssim d\log\left(\frac{1}{h\varepsilon}\right)\lesssim\log n+\log(1/\varepsilon),\qquad 0<\varepsilon\le 1.$$

Dudley's entropy integral therefore gives

$$\mathbb{E}\left[\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|\right]\lesssim\sqrt{\log n}.$$

Choosing $a_n\asymp(\mathfrak{R}_{\mathtt{sa}}/\sqrt{\log n})^{1/2}$ gives the result. □

Lemma SA-14 (KS Distance Between $\widehat{\mathrm{T}}$ and $\overline{\mathrm{T}}$)**.** *Suppose the conditions of Theorem SA-8 hold. Then*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}(\mathbf{x})|\le u\right)-\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\overline{\mathrm{T}}(\mathbf{x})|\le u\right)\right|=o(1).$$

Proof of Lemma SA-14. Theorem SA-6 gives

$$\Delta_{\widehat{T},\overline{T}}:=\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}(\mathbf{x})-\overline{\mathrm{T}}(\mathbf{x})|=O_{\mathbb{P}}(\mathfrak{R}_{\mathtt{lin}}).$$

The side conditions imply $\mathfrak{R}_{\mathtt{lin}}\sqrt{\log n}=o(1)$. Choose a deterministic sequence a_n such that $\mathfrak{R}_{\mathtt{lin}}=o(a_n)$ and $a_n\sqrt{\log n}=o(1)$. Applying Lemma SA-12 with $\mathbb{T}_1=\widehat{\mathrm{T}}$ and $\mathbb{T}_2=\overline{\mathrm{T}}$ gives

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}(M_{\widehat{T}}\le u)-\mathbb{P}(M_{\overline{T}}\le u)\right|\le\sup_{u\in\mathbb{R}}\mathbb{P}(|M_{\overline{T}}-u|\le a_n)+\mathbb{P}(\Delta_{\widehat{T},\overline{T}}>a_n),$$

where $M_{\widehat{T}}=\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}(\mathbf{x})|$ and $M_{\overline{T}}=\sup_{\mathbf{x}\in\mathscr{B}}|\overline{\mathrm{T}}(\mathbf{x})|$. The second term is $o(1)$. The anti-concentration term for $M_{\overline{T}}$ is bounded through Lemma SA-13: for $M_Z=\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|$,

$$\mathbb{P}(|M_{\overline{T}}-u|\le a_n)\le\mathbb{P}(|M_Z-u|\le a_n)+2\sup_{v\in\mathbb{R}}|\mathbb{P}(M_{\overline{T}}\le v)-\mathbb{P}(M_Z\le v)|$$

$$\leq 4a_n\{\mathbb{E}[M_Z]+1\} + 2\sup_{v\in\mathbb{R}}|\mathbb{P}(M_{\overline{T}}\leq v) - \mathbb{P}(M_Z\leq v)|.$$

The first term is $o(1)$ because $a_n\sqrt{\log n} = o(1)$, and the second is $o(1)$ by Lemma SA-13. This proves the result. □

Lemma SA-15 (KS Distance Between Z and $\widehat{Z}$)**.** *Suppose the conditions of Theorem SA-8 hold. Then*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|Z(\mathbf{x})|\leq u\right) - \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{Z}(\mathbf{x})|\leq u \mid \mathbf{V}\right)\right| \lesssim_{\mathbb{P}} \log(n)\mathfrak{R}_{\mathsf{cc}}^{1/2} = o_{\mathbb{P}}(1).$$

Proof of Lemma SA-15. Let

$$\Pi_{\mathbf{x},\mathbf{y}} = \frac{\Xi_{\mathbf{x},\mathbf{y}}}{\sqrt{\Xi_{\mathbf{x},\mathbf{x}}\Xi_{\mathbf{y},\mathbf{y}}}}, \qquad \widehat{\Pi}_{\mathbf{x},\mathbf{y}} = \frac{\widehat{\Xi}_{\mathbf{x},\mathbf{y}}}{\sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}\widehat{\Xi}_{\mathbf{y},\mathbf{y}}}}.$$

Lemma SA-5 gives

$$\sup_{\mathbf{x},\mathbf{y}\in\mathscr{B}}|\widehat{\Pi}_{\mathbf{x},\mathbf{y}} - \Pi_{\mathbf{x},\mathbf{y}}| \lesssim_{\mathbb{P}} \mathfrak{R}_{\mathsf{cc}},$$

because $\inf_{\mathbf{x}\in\mathscr{B}}\Xi_{\mathbf{x},\mathbf{x}} \gtrsim (nh^d)^{-1}$ and $\inf_{\mathbf{x}\in\mathscr{B}}\widehat{\Xi}_{\mathbf{x},\mathbf{x}} \gtrsim_{\mathbb{P}} (nh^d)^{-1}$.

Let $\mathscr{B}_{\delta_n}$ be a δ_n-net of $\mathscr{B}$ under $\|\cdot\|_\infty$ with cardinality $M \lesssim \delta_n^{-d}$, and write $\mathbf{Z}^{\delta_n}$ and $\widehat{\mathbf{Z}}^{\delta_n}$ for the restrictions of Z and $\widehat{Z}$ to this net. By Chernozhukov et al. [2022, Theorem 2.1],

$$\sup_{\mathbf{y}\in\mathbb{R}^M}\left|\mathbb{P}(\mathbf{Z}^{\delta_n}\leq\mathbf{y}) - \mathbb{P}(\widehat{\mathbf{Z}}^{\delta_n}\leq\mathbf{y}\mid\mathbf{V})\right| \lesssim_{\mathbb{P}} \log(M)\mathfrak{R}_{\mathsf{cc}}^{1/2}.$$

Hence the same bound applies to the distribution functions of $\|\mathbf{Z}^{\delta_n}\|_\infty$ and $\|\widehat{\mathbf{Z}}^{\delta_n}\|_\infty$.

Choose $\delta_n = n^{-s}$ with $s>0$ large enough that

$$\sqrt{\log n}\, h^{-1}\delta_n = o(\mathfrak{R}_{\mathsf{cc}}^{1/2}).$$

Such an s exists because h is bounded below by a polynomial in n. The same Gaussian-metric calculation used in Lemma SA-13 and its feasible analogue imply

$$\mathbb{E}\left[\sup_{\substack{\mathbf{x},\mathbf{y}\in\mathscr{B}\\ \|\mathbf{x}-\mathbf{y}\|_\infty\leq\delta_n}}|Z(\mathbf{x})-Z(\mathbf{y})|\right] \lesssim \sqrt{\log n}\, h^{-1}\delta_n,$$

$$\mathbb{E}\left[\sup_{\substack{\mathbf{x},\mathbf{y}\in\mathscr{B}\\ \|\mathbf{x}-\mathbf{y}\|_\infty\leq\delta_n}}|\widehat{Z}(\mathbf{x})-\widehat{Z}(\mathbf{y})|\,\middle|\,\mathbf{V}\right] \lesssim_{\mathbb{P}} \sqrt{\log n}\, h^{-1}\delta_n.$$

Markov's inequality makes both modulus errors $o_{\mathbb{P}}(\mathfrak{R}_{\mathsf{cc}}^{1/2})$. Combining the finite-net comparison, the two modulus bounds, and Gaussian anti-concentration gives the displayed Kolmogorov bound. The side conditions imply $\log(n)\mathfrak{R}_{\mathsf{cc}}^{1/2} = o_{\mathbb{P}}(1)$. □

The proof of Theorem SA-8 now follows from the triangle inequality applied to the three comparison lemmas above. Furthermore, by definition of $\widehat{\mathrm{I}}_\alpha(\mathbf{x})$,

$$\mathbb{P}\Big[\vartheta(\mathbf{x})\in\widehat{\mathrm{I}}_\alpha(\mathbf{x}),\ \text{for all } \mathbf{x}\in\mathscr{B}\Big] = \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}(\mathbf{x})|\leq \mathfrak{q}_\alpha\right)$$

$$= \mathbb{E}\left[\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{Z}(\mathbf{x})| \le \mathfrak{q}_\alpha \mid \mathbf{V}\right)\right] + o(1)$$
$$= 1 - \alpha + o(1).$$

This completes the proof. □

SA-8.14 Proof of Theorem SA-9

We employ the following result established by Chernozhukov et al. [2014b, Corollary 2.2].

Lemma SA-16 (Gaussian Approximation of Suprema)**.** *Let $X_1, \ldots, X_n$ be i.i.d. random variables in a measurable space $(S, \mathcal{S})$ with law P, where $n \ge 3$. Let $\mathscr{F}$ be a class of P-centered measurable functions $f : S \to \mathbb{R}$, and define $\mathbb{G}_n f = n^{-1/2}\sum_{i=1}^n f(X_i)$. Suppose that:*

(i) *$\mathscr{F}$ is pointwise measurable.*

(ii) *There exists a measurable envelope F such that $|f| \le F$ for all $f \in \mathscr{F}$, and constants $A \ge e$ and $\upsilon \ge 1$ such that*

$$\sup_Q N(\mathscr{F}, \|\cdot\|_{Q,2}, \epsilon\|F\|_{Q,2}) \le (A/\epsilon)^\upsilon, \qquad 0 < \epsilon \le 1,$$

where the supremum is over all finite discrete probability measures on $(S, \mathcal{S})$.

(iii) *For some $b \ge \sigma > 0$ and $q \in [4, \infty]$,*

$$\sup_{f\in\mathscr{F}} P|f|^k \le \sigma^2 b^{k-2}, \qquad k = 2, 3, 4, \qquad \text{and} \qquad \|F\|_{P,q} \le b.$$

Let $Z = \sup_{f\in\mathscr{F}} \mathbb{G}_n f$. Then, for every $\gamma \in (0, 1)$, there exists, on a possibly enlarged probability space, a tight mean-zero Gaussian process $(\mathbb{G}_P f : f \in \mathscr{F})$ with covariance function $\mathbb{E}[\mathbb{G}_P(f)\mathbb{G}_P(g)] = P(fg)$ and almost surely uniformly $\|\cdot\|_{P,2}$-continuous trajectories such that, for $\widetilde{Z} = \sup_{f\in\mathscr{F}} \mathbb{G}_P f$,

$$\mathbb{P}\left\{|Z - \widetilde{Z}| > \frac{bK_n}{\gamma^{1/q} n^{1/2-1/q}} + \frac{(b\sigma)^{1/2}K_n^{3/4}}{\gamma^{1/2}n^{1/4}} + \frac{(b\sigma^2K_n^2)^{1/3}}{\gamma^{1/3}n^{1/6}}\right\} \le C\left(\gamma + \frac{\log n}{n}\right),$$

where $K_n = \upsilon(\log n \vee \log(Ab/\sigma))$, and $C > 0$ depends only on q.

Let $\mathbf{z}_i = (\mathbf{X}_i, Y_i)$, $1 \le i \le n$. For each $\mathbf{x} \in \mathscr{B}$, define

$$\tilde{f}_\mathbf{x}(\mathbf{u}, u) = g_\mathbf{x}(\mathbf{u})u + m_\mathbf{x}(\mathbf{u}), \qquad f_\mathbf{x}(\mathbf{u}, u) = \tilde{f}_\mathbf{x}(\mathbf{u}, u) - \mathbb{E}[\tilde{f}_\mathbf{x}(\mathbf{X}_i, Y_i)].$$

Let $\mathscr{F}_\pm = \{f_\mathbf{x} : \mathbf{x} \in \mathscr{B}\} \cup \{-f_\mathbf{x} : \mathbf{x} \in \mathscr{B}\}$. By the representation preceding Theorem SA-7,

$$\sup_{\mathbf{x}\in\mathscr{B}} |\overline{\mathrm{T}}(\mathbf{x})| = \sup_{f\in\mathscr{F}_\pm} \frac{1}{\sqrt{n}}\sum_{i=1}^n f(\mathbf{z}_i).$$

We apply Lemma SA-16 to $\mathscr{F}_\pm$.

First, $\mathscr{F}_\pm$ is pointwise measurable by the compactness and separability of $\mathscr{B}$ and the measurability of the signed-distance kernels. The entropy calculation used in the proof of Theorem SA-7, but without the total-variation step, gives a VC-type bound for $\mathscr{G}$ and $\mathscr{M}$. The fixed side indicators $\mathbb{1}(\mathbf{u} \in \mathscr{A}_t)$ only contract

$L_2(Q)$ distances. Thus, with envelope

$$F(\mathbf{u}, u) = Ch^{-d/2}(|u| + 1),$$

for a sufficiently large constant C, there are constants $A_h \geq e$ and $\upsilon \geq 1$, with A_h at most polynomial in $1/h$, such that

$$\sup_Q N(\mathscr{F}_\pm, \|\cdot\|_{Q,2}, \epsilon\|F\|_{Q,2}) \leq (A_h/\epsilon)^\upsilon, \qquad 0 < \epsilon \leq 1.$$

Centering changes the covering constants only by a universal factor because $|P(\tilde{f}_\mathbf{x} - \tilde{f}_\mathbf{y})| \leq \|\tilde{f}_\mathbf{x} - \tilde{f}_\mathbf{y}\|_{P,2}$.

For the moment condition, take $q = 2 + \upsilon \geq 4$. Assumption SA-1(v) and the boundedness of $\theta^*_{t,\mathbf{x}}(D_\mathbf{u}(\mathbf{x}))$ on the kernel support give

$$\|F\|_{P,2+\upsilon} \lesssim h^{-d/2}.$$

The same local-polynomial moment calculation used in the proofs of Lemmas SA-4 and SA-5 gives

$$\sup_{f \in \mathscr{F}_\pm} P|f|^k \lesssim h^{-d(k-2)/2}, \qquad k = 2, 3, 4.$$

Hence Lemma SA-16 applies with $\sigma = 1$ and $b \asymp h^{-d/2}$. Since $\liminf_{n\to\infty} \log h / \log n > -\infty$ and A_h is at most polynomial in $1/h$, $K_n \lesssim \log n$. Taking $\gamma = (\log n)^{-1}$ yields

$$\mathbb{P}\left[\left|\sup_{\mathbf{x}\in\mathscr{B}} |\overline{\mathrm{T}}(\mathbf{x})| - \sup_{\mathbf{x}\in\mathscr{B}} |Z(\mathbf{x})|\right| > \frac{\log n}{(nh^d)^{1/6}} + \frac{(\log n)^{5/4}}{(nh^d)^{1/4}} + \frac{(\log n)^{3/2}}{(n^{\frac{\upsilon}{2+\upsilon}} h^d)^{1/2}}\right] \lesssim \frac{1}{\log n} + \frac{\log n}{n}.$$

The covariance of the Gaussian process delivered by Lemma SA-16 is $P(f_{\mathbf{x}_1} f_{\mathbf{x}_2})$, which equals $\mathbb{C}\mathrm{ov}[\overline{\mathrm{T}}(\mathbf{x}_1), \overline{\mathrm{T}}(\mathbf{x}_2)]$. This proves the theorem. □

SA-8.15 Proof of Theorem SA-10

Define

$$\mathfrak{R}_{\mathtt{sup}} = \frac{\log n}{(nh^d)^{1/6}} + \frac{(\log n)^{5/4}}{(nh^d)^{1/4}} + \frac{(\log n)^{3/2}}{(n^{\frac{\upsilon}{2+\upsilon}} h^d)^{1/2}}.$$

The entropy calculation in the proof of Theorem SA-9 implies

$$\mathbb{E}\left[\sup_{\mathbf{x}\in\mathscr{B}} |Z(\mathbf{x})|\right] \lesssim \sqrt{\log n}.$$

The condition $n^{\frac{\upsilon}{2+\upsilon}} h^d/(\log n)^9 \to \infty$ implies $\mathfrak{R}_{\mathtt{sup}}\sqrt{\log n} = o(1)$. Let

$$M_{\overline{T}} = \sup_{\mathbf{x}\in\mathscr{B}} |\overline{\mathrm{T}}(\mathbf{x})|, \qquad M_Z = \sup_{\mathbf{x}\in\mathscr{B}} |Z(\mathbf{x})|.$$

Theorem SA-9 and Gaussian anti-concentration give

$$\sup_{u\in\mathbb{R}} |\mathbb{P}(M_{\overline{T}} \le u) - \mathbb{P}(M_Z \le u)| = o(1). \tag{SA-23}$$

Next, Theorem SA-6 gives

$$\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{\mathrm{T}}(\mathbf{x}) - \overline{\mathrm{T}}(\mathbf{x})| = O_{\mathbb{P}}(\mathfrak{R}_{\mathtt{lin}}),$$

where

$$\mathfrak{R}_{\mathtt{lin}} = \sqrt{\log(1/h)} \left(\sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}} h^d} \right) + \sqrt{nh^d} \sup_{\mathbf{x}\in\mathscr{B}} |\mathfrak{B}(\mathbf{x})|.$$

The polynomial lower bound on h, the condition $n^{\frac{v}{2+v}} h^d/(\log n)^9 \to \infty$, and the stated small-bias condition imply $\mathfrak{R}_{\mathtt{lin}}\sqrt{\log n} = o(1)$. Combining this with Equation (SA-23) and another Gaussian anti-concentration step yields

$$\sup_{u\in\mathbb{R}} \left| \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{\mathrm{T}}(\mathbf{x})| \le u \right) - \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |\overline{\mathrm{T}}(\mathbf{x})| \le u \right) \right| = o(1). \tag{SA-24}$$

Finally, Lemma SA-5 gives the feasible covariance-comparison rate

$$\mathfrak{R}_{\mathtt{cc}} = \sqrt{\frac{\log(1/h)}{nh^d}} + \frac{\log(1/h)}{n^{\frac{v}{2+v}} h^d}.$$

The same finite-net Gaussian comparison and anti-concentration argument used in the proof of Theorem SA-8 gives

$$\sup_{u\in\mathbb{R}} \left| \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |Z(\mathbf{x})| \le u \right) - \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{Z}(\mathbf{x})| \le u \mid \mathbf{V} \right) \right| \lesssim_{\mathbb{P}} \log(n)\mathfrak{R}_{\mathtt{cc}}^{1/2} = o_{\mathbb{P}}(1), \tag{SA-25}$$

where the last equality follows from $n^{\frac{v}{2+v}} h^d/(\log n)^9 \to \infty$. The displayed Kolmogorov approximation follows from the triangle inequality applied to Equations (SA-23), (SA-24), and (SA-25).

The simultaneous coverage statement follows from the definition of $\widehat{\mathrm{I}}_\alpha(\mathbf{x})$ and the conditional critical value $\mathfrak{q}_\alpha$:

$$\begin{aligned}
\mathbb{P}\Big[\vartheta(\mathbf{x}) \in \widehat{\mathrm{I}}_\alpha(\mathbf{x}),\ \text{for all } \mathbf{x}\in\mathscr{B}\Big] &= \mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{\mathrm{T}}(\mathbf{x})| \le \mathfrak{q}_\alpha \right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}} |\widehat{Z}(\mathbf{x})| \le \mathfrak{q}_\alpha \mid \mathbf{V} \right) \right] + o(1) \\
&= 1 - \alpha + o(1).
\end{aligned}$$

□

SA-8.16 Proof of Lemma SA-6

Let $\mathfrak{L}_{t,\mathtt{WBATE}}$ and $\Xi_{t,\mathtt{WBATE}}$ denote the side-specific analogues of $\mathfrak{L}_{\mathtt{WBATE}}$ and $\Xi_{\mathtt{WBATE}}$. Fubini's theorem and the definition of $\Xi_{t,\mathbf{x}_1,\mathbf{x}_2}$ give

$$\mathbb{V}\left(\mathfrak{L}_{t,\mathtt{WBATE}}\right) = \Xi_{t,\mathtt{WBATE}}.$$

The covariance between the two side-specific linear terms is zero because the treatment and control assignment indicators are disjoint observation by observation. Hence $\mathbb{V}(\mathfrak{L}_{\mathtt{WBATE}}) = \Xi_{\mathtt{WBATE}}$.

Let $R_K < \infty$ satisfy $\operatorname{supp}(K) \subseteq [-R_K, R_K]$. By equivalence of $\mathscr{d}$ and the Euclidean distance, if $\Xi_{\mathbf{x}_1,\mathbf{x}_2} \neq 0$ then $\|\mathbf{x}_1 - \mathbf{x}_2\| \lesssim h$. Therefore

$$\mathscr{E}(h) = \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathscr{B}^2 : \|\mathbf{x}_1 - \mathbf{x}_2\| \leq Ch\}$$

has product boundary measure $\nu_{\mathscr{B},2}(\mathscr{E}(h)) \lesssim h^{d-1}$ by (SA-4). Uniform boundedness of $\mathbf{\Psi}_{t,\mathbf{x}}^{-1}$ and $\mathbf{\Upsilon}_{t,\mathbf{x}_1,\mathbf{x}_2}$ gives $|\Xi_{\mathbf{x}_1,\mathbf{x}_2}| \lesssim (nh^d)^{-1}$, and hence

$$\Xi_{\mathtt{WBATE}} \lesssim \frac{h^{d-1}}{nh^d} = \frac{1}{nh}.$$

Under (SA-5), this upper bound is complemented by the matching lower bound.

The same support restriction applies to $\widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_2}$. Hence, using Lemma SA-5,

$$\begin{aligned}
\left|\widehat{\Xi}_{\mathtt{WBATE}} - \Xi_{\mathtt{WBATE}}\right| &\leq \int_{\mathscr{E}(h)} \left|\widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_2} - \Xi_{\mathbf{x}_1,\mathbf{x}_2}\right| |w(\mathbf{x}_1)w(\mathbf{x}_2)| \, \mathrm{d}\nu_{\mathscr{B},2}(\mathbf{x}_1,\mathbf{x}_2) \\
&\lesssim_{\mathbb{P}} h^{d-1} \sup_{\mathbf{x}_1,\mathbf{x}_2 \in \mathscr{B}} \left|\widehat{\Xi}_{\mathbf{x}_1,\mathbf{x}_2} - \Xi_{\mathbf{x}_1,\mathbf{x}_2}\right| \\
&= o_{\mathbb{P}}\Big(\frac{1}{nh}\Big),
\end{aligned}$$

which proves the plug-in variance claim. The ratio claim follows by dividing this additive bound by the lower bound in (SA-5). □

SA-8.17 Proof of Lemma SA-7

It is enough to consider one side $t \in \{0, 1\}$. Define the side-specific nonlinear WBATE remainder

$$\mathfrak{Q}_{t,\mathtt{WBATE}} = \int_{\mathscr{B}} \mathbf{e}_0^\top (\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1} - \mathbf{\Psi}_{t,\mathbf{x}}^{-1}) \mathbf{O}_{t,\mathbf{x}} w(\mathbf{x}) \, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}), \qquad \mathfrak{Q}_{\mathtt{WBATE}} = \mathfrak{Q}_{1,\mathtt{WBATE}} - \mathfrak{Q}_{0,\mathtt{WBATE}}.$$

Let

$$\mathbf{A}_{t,\mathbf{x}} = \widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1} - \mathbf{\Psi}_{t,\mathbf{x}}^{-1} = -\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1} (\widehat{\mathbf{\Psi}}_{t,\mathbf{x}} - \mathbf{\Psi}_{t,\mathbf{x}}) \mathbf{\Psi}_{t,\mathbf{x}}^{-1}.$$

On the event where $\sup_{\mathbf{x} \in \mathscr{B}} \|\widehat{\mathbf{\Psi}}_{t,\mathbf{x}}^{-1}\| \lesssim 1$, which has probability approaching one by Lemma SA-3,

$$|\mathfrak{Q}_{t,\mathtt{WBATE}}| \lesssim \left| \int_{\mathscr{B}} \mathbf{A}_{t,\mathbf{x}} \mathbf{O}_{t,\mathbf{x}} w(\mathbf{x}) \, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}) \right|.$$

We suppress fixed finite-dimensional vectors and matrices in the bounds below.

Decompose $\mathbf{O}_{t,\mathbf{x}}$ into a conditional-mean-zero part and a conditional-mean part:

$$\mathbf{O}_{t,\mathbf{x}} = \mathbf{Q}_{t,\mathbf{x}} + \mathbf{M}_{t,\mathbf{x}},$$

where

$$\mathbf{Q}_{t,\mathbf{x}} = \mathbb{E}_n\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)K_h(D_i(\mathbf{x}))\varepsilon_{t,i}\mathbb{1}(\mathbf{X}_i \in \mathscr{A}_t)\right],$$
$$\mathbf{M}_{t,\mathbf{x}} = \mathbb{E}_n\left[\mathbf{r}_p\Big(\frac{D_i(\mathbf{x})}{h}\Big)K_h(D_i(\mathbf{x}))\{\mu_t(\mathbf{X}_i) - \theta^*_{t,\mathbf{x}}(D_i(\mathbf{x}))\}\mathbb{1}(\mathbf{X}_i \in \mathscr{A}_t)\right],$$

with $\varepsilon_{t,i} = Y_i(t) - \mu_t(\mathbf{X}_i)$. The first component is mean zero conditional on $\mathbf{X}_1, \ldots, \mathbf{X}_n$. The second component is not conditionally mean zero; this is the additional distance-based term absent from the location-based analysis.

For the first component, conditional on $\mathbf{X}_1, \ldots, \mathbf{X}_n$, write $\psi_{t,\mathbf{x}}(\mathbf{z}) = \mathbf{r}_p(D_{\mathbf{z}}(\mathbf{x})/h)K_h(D_{\mathbf{z}}(\mathbf{x}))\mathbb{1}(\mathbf{z} \in \mathscr{A}_t)$. The summand inside the boundary integral is nonzero only when $\mathbf{z}$ is within an $O(h)$ neighborhood of $\mathbf{x}$. Therefore, after applying Fubini to the conditional second moment, the only pairs $(\mathbf{x}_1, \mathbf{x}_2)$ that contribute are those satisfying $\|\mathbf{x}_1 - \mathbf{x}_2\| \lesssim h$. The local boundary-measure condition (SA-4) gives product Hausdorff measure $O(h^{d-1})$ for this effective pair set, while each localized kernel product has conditional expectation of order h^{-d}. Hence

$$\mathbb{E}\left[\left|\int_{\mathscr{B}} \mathbf{A}_{t,\mathbf{x}}\mathbf{Q}_{t,\mathbf{x}} w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})\right|^2 \middle| \mathbf{X}_1, \ldots, \mathbf{X}_n\right] \lesssim \frac{\sup_{\mathbf{x}\in\mathscr{B}}\|\mathbf{A}_{t,\mathbf{x}}\|^2}{nh}.$$

Lemma SA-3 therefore implies

$$\int_{\mathscr{B}} \mathbf{A}_{t,\mathbf{x}}\mathbf{Q}_{t,\mathbf{x}} w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}) = O_{\mathbb{P}}\left(\frac{\sqrt{\log(1/h)}}{nh^{(d+1)/2}}\right) = o_{\mathbb{P}}\Big(\frac{1}{nh^d}\Big),$$

because $d \geq 2$, $h = o(1)$, and $h^{d-1}\log(1/h) \to 0$.

It remains to bound the conditional-mean component. Write each coordinate of $\int_{\mathscr{B}} \mathbf{A}_{t,\mathbf{x}}\mathbf{M}_{t,\mathbf{x}} w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x})$ as

$$\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n H_{ij}, \qquad H_{ij} = \int_{\mathscr{B}} a_{\mathbf{x}}(\mathbf{X}_i)m_{\mathbf{x}}(\mathbf{X}_j)w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}),$$

where $a_{\mathbf{x}}$ is a centered coordinate of $\widehat{\mathbf{\Psi}}_{t,\mathbf{x}} - \mathbf{\Psi}_{t,\mathbf{x}}$ and $m_{\mathbf{x}}$ is the corresponding conditional-mean coordinate of $\mathbf{M}_{t,\mathbf{x}}$. The localized part of $a_{\mathbf{x}}$ and the whole of $m_{\mathbf{x}}$ are supported on $O(h)$-neighborhoods of $\mathbf{x}$, with envelopes of order h^{-d}. The centering constant in $a_{\mathbf{x}}$ is only $O(1)$ and therefore gives smaller contributions than the localized part. Thus the dominant part of H_{ij} is nonzero only when the relevant observations are within distance $O(h)$ of the same boundary point.

For the diagonal term, the boundary integral has size at most $h^{-2d}\mathfrak{H}^{d-1}(\mathscr{B} \cap B(\mathbf{X}_i, Ch)) \lesssim h^{-d-1}$ on the event that $\mathbf{X}_i$ lies in an $O(h)$ tube around $\mathscr{B}$, and is zero otherwise up to the smaller centering-constant contribution. Since this tube has probability $O(h)$,

$$\mathbb{E}|H_{ii}| \lesssim h^{-d}.$$

For the off-diagonal term, independence and Fubini give

$$\mathbb{E}[H_{ij}^2] = \int_{\mathscr{B}}\int_{\mathscr{B}} \mathbb{E}[a_{\mathbf{x}_1}(\mathbf{X}_i)a_{\mathbf{x}_2}(\mathbf{X}_i)]\mathbb{E}[m_{\mathbf{x}_1}(\mathbf{X}_j)m_{\mathbf{x}_2}(\mathbf{X}_j)]w(\mathbf{x}_1)w(\mathbf{x}_2)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_1)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_2).$$

The leading localized contribution to the two expectations in the integrand is zero unless $\|\mathbf{x}_1 - \mathbf{x}_2\| \lesssim h$, and the terms containing the centering constants are of the same order or smaller. On this effective set each expectation is bounded by $O(h^{-d})$, and the effective set has product Hausdorff measure $O(h^{d-1})$. Hence

$$\mathbb{E}[H_{ij}^2] \lesssim h^{-d-1}, \qquad i \neq j.$$

The off-diagonal kernel is degenerate because $\mathbb{E}[a_{\mathbf{x}}(\mathbf{X}_i)] = 0$ and $\mathbb{E}[m_{\mathbf{x}}(\mathbf{X}_j)] = 0$ for every $\mathbf{x}$. Thus,

$$\frac{1}{n^2}\sum_{i=1}^{n} H_{ii} = O_{\mathbb{P}}\Big(\frac{1}{nh^d}\Big), \qquad \frac{1}{n^2}\sum_{i\neq j} H_{ij} = O_{\mathbb{P}}\Big(\frac{1}{nh^{(d+1)/2}}\Big) = o_{\mathbb{P}}\Big(\frac{1}{nh^d}\Big).$$

Combining the two components and summing over $t \in \{0,1\}$ gives $\mathfrak{Q}_{\mathtt{WBATE}} = O_{\mathbb{P}}(1/(nh^d))$. Finally,

$$\frac{1/(nh^d)}{1/\sqrt{nh}} = \frac{1}{\sqrt{nh^{2d-1}}},$$

which is $o(1)$ under $nh^{2d-1} \to \infty$. □

SA-8.18 Proof of Theorem SA-11

The decomposition (SA-3), Lemma SA-6, and Lemma SA-7 give

$$\Big|\widehat{\vartheta}_{\mathtt{WBATE}} - \vartheta_{\mathtt{WBATE}}\Big| \leq |\mathfrak{B}_{\mathtt{WBATE}}| + O_{\mathbb{P}}\Big(\frac{1}{\sqrt{nh}}\Big) + O_{\mathbb{P}}\Big(\frac{1}{nh^d}\Big),$$

which is the stated rate. □

SA-8.19 Proof of Theorem SA-12

By (SA-3), Lemma SA-7, the small-bias condition, and the aggregate nondegeneracy condition,

$$\frac{\widehat{\vartheta}_{\mathtt{WBATE}} - \vartheta_{\mathtt{WBATE}} - \mathfrak{L}_{\mathtt{WBATE}}}{\sqrt{\Xi_{\mathtt{WBATE}}}} = o_{\mathbb{P}}(1).$$

Lemma SA-6 gives $\widehat{\Xi}_{\mathtt{WBATE}}/\Xi_{\mathtt{WBATE}} \to_{\mathbb{P}} 1$.

Write $\mathfrak{L}_{\mathtt{WBATE}} = \sum_{i=1}^{n} \ell_i$, where the ℓ_i are independent, mean-zero summands obtained by integrating the side-specific linear influence functions over $\mathscr{B}$. By construction, $\sum_{i=1}^{n}\mathbb{E}[\ell_i^2] = \Xi_{\mathtt{WBATE}}$. To verify Lyapunov's condition, let

$$\mathscr{E}_3(h) = \{(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \in \mathscr{B}^3 : \|\mathbf{x}_j - \mathbf{x}_k\| \leq Ch,\ 1 \leq j,k \leq 3\}.$$

Compact support of K, metric equivalence, boundedness of w, and (SA-4) imply $\nu_{\mathscr{B},3}(\mathscr{E}_3(h)) \lesssim h^{2d-2}$. The

same local-moment calculation used in Lemma SA-4 gives

$$\sum_{i=1}^{n}\mathbb{E}\left[\left|\frac{\ell_i}{\sqrt{\Xi_{\mathtt{WBATE}}}}\right|^3\right]\lesssim\frac{1}{\sqrt{nh}}\to 0,$$

where the last step follows from $nh^{2d-1}\to\infty$ and $d\geq 2$. Lyapunov's central limit theorem gives

$$\frac{\mathfrak{L}_{\mathtt{WBATE}}}{\sqrt{\Xi_{\mathtt{WBATE}}}}\rightsquigarrow\mathsf{Normal}(0,1).$$

Slutsky's theorem gives the feasible statistic result, and the confidence interval statement follows from the standard normal critical value. □

SA-8.20 Proof of Theorem SA-13

The supremum functional is one-Lipschitz under the sup-norm:

$$\left|\sup_{\mathbf{x}\in\mathscr{B}}\widehat{\vartheta}(\mathbf{x})-\sup_{\mathbf{x}\in\mathscr{B}}\vartheta(\mathbf{x})\right|\leq\sup_{\mathbf{x}\in\mathscr{B}}\left|\widehat{\vartheta}(\mathbf{x})-\vartheta(\mathbf{x})\right|.$$

The theorem follows immediately from Theorem SA-4. □

SA-8.21 Proof of Theorem SA-14

On the event

$$\vartheta(\mathbf{x})\in\left[\widehat{\vartheta}(\mathbf{x})-\mathfrak{q}_\alpha\sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}},\widehat{\vartheta}(\mathbf{x})+\mathfrak{q}_\alpha\sqrt{\widehat{\Xi}_{\mathbf{x},\mathbf{x}}}\right]\quad\text{for all }\mathbf{x}\in\mathscr{B},$$

taking suprema over $\mathbf{x}\in\mathscr{B}$ implies

$$\vartheta_{\mathtt{LBATE}}\in\widehat{\mathrm{I}}_{\alpha,\mathtt{LBATE}}.$$

Theorem SA-10 gives the first event probability $1-\alpha+o(1)$, which proves the result. □

SA-8.22 Proof of Theorem SA-15

For $A=Y$, Theorem SA-1 applies directly after the reduced-form redefinition $Y_i(t)=Y_i(t,W_i(t))$. For $A=W$, the same proof applies with $W_i(t)$ in place of $Y_i(t)$. Thus the sharp-design identification argument is applied separately to the reduced-form outcome component and the first-stage treatment-receipt component: Assumption SA-4(i)–(ii) gives the reduced-form smoothness and sampling conditions, while Assumption SA-2(i), (ii), and (v) gives the same distance-based identification conditions. Therefore $\vartheta_A(\mathbf{x})=\tau_A(\mathbf{x})$ for $A\in\{Y,W\}$. If $\tau_W(\mathbf{x})\neq 0$, then the ratio identity $\xi(\mathbf{x})=\zeta(\mathbf{x})$ follows immediately. □

SA-8.23 Proof of Theorem SA-16

Theorem SA-4 applies to the reduced-form outcome estimator $\widehat{\vartheta}_Y(\mathbf{x})$. The same proof applies to $\widehat{\vartheta}_W(\mathbf{x})$ because Assumption SA-4(i)–(ii) gives the corresponding smoothness and moment conditions; boundedness

of $W_i(t)$ gives the required moment bounds automatically. Thus, pointwise and uniformly,

$$\widehat{\vartheta}_A(\mathbf{x}) - \vartheta_A(\mathbf{x}) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{nh^d}} + \frac{1}{n^{\frac{1+v}{2+v}}h^d} + |\mathfrak{B}_A(\mathbf{x})|\right)$$

with the uniform version replacing the first two terms by their logarithmic analogues. Because $v \geq 2$, the pointwise condition $(n^{(1+v)/(2+v)}h^d)^{-1} = o(1)$ implies $(nh^d)^{-1/2} = o(1)$. Uniformly, $\log(1/h)/(n^{(1+v)/(2+v)}h^d) = o(1)$ implies the corresponding logarithmic stochastic terms are $o(1)$. The condition $h = o(1)$ supplies the shrinking-neighborhood requirement in the reduced-form distance-based rates, and Theorem SA-2 implies the reduced-form approximation biases are $o(1)$. Together, these restrictions imply that $\widehat{\vartheta}_W(\mathbf{x})$ is bounded away from zero with probability approaching one, pointwise or uniformly as appropriate. The exact Wald expansion (SA-7) and the bound (SA-8) then give the stated rates. □

SA-8.24 Proof of Lemma SA-8

Fix $A, B \in \{Y, W\}$. The proof of Lemma SA-5 applies to the cross-covariance kernel $\Xi_{A,B,\mathbf{x}_1,\mathbf{x}_2}$ with

$$(A_i - \theta^*_{A,t,\mathbf{x}_1}(D_i(\mathbf{x}_1)))(B_i - \theta^*_{B,t,\mathbf{x}_2}(D_i(\mathbf{x}_2)))$$

in place of the squared residual product. The moment bounds needed for $A = B = Y$ follow from Assumption SA-1(v); those involving W follow from $W_i(t) \in \{0,1\}$ and Cauchy–Schwarz. Assumption SA-4(iii) provides continuity of the reduced-form conditional covariance entries. Hence, uniformly over $\mathbf{x}_1, \mathbf{x}_2 \in \mathscr{B}$,

$$\sup_{A,B\in\{Y,W\}} nh^d\left|\widehat{\Xi}_{A,B,\mathbf{x}_1,\mathbf{x}_2} - \Xi_{A,B,\mathbf{x}_1,\mathbf{x}_2}\right| = o_{\mathbb{P}}(1).$$

Theorem SA-16 gives $\sup_{\mathbf{x}\in\mathscr{B}} \|\widehat{\mathfrak{w}}(\mathbf{x}) - \mathfrak{w}(\mathbf{x})\| = o_{\mathbb{P}}(1)$ under the denominator condition and the stated bandwidth restrictions. Combining the last two displays with the definitions of $\Xi_{\mathtt{F}}$ and $\widehat{\Xi}_{\mathtt{F}}$ gives

$$\sup_{\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}} nh^d|\widehat{\Xi}_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2} - \Xi_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}| = o_{\mathbb{P}}(1).$$

The relative consistency statement follows from the imposed lower bound $\inf_{\mathbf{x}\in\mathscr{B}} nh^d\Xi_{\mathtt{F},\mathbf{x},\mathbf{x}} \gtrsim 1$. □

SA-8.25 Proof of Theorem SA-17

Because $v \geq 2$, the condition $n^{v/(2+v)}h^d \to \infty$ implies both $nh^d \to \infty$ and $(n^{(1+v)/(2+v)}h^d)^{-1} \to 0$. Therefore the reduced-form stochastic terms for Y and W are consistent. The condition $h = o(1)$ gives shrinking local neighborhoods, and Theorem SA-2 makes the first-stage approximation bias vanish.

For the fixed $\mathbf{x}$, the exact expansion (SA-7) gives

$$\widehat{\xi}(\mathbf{x}) - \xi(\mathbf{x}) = \mathfrak{B}_{\mathtt{F}}(\mathbf{x}) + \mathfrak{L}_{\mathtt{F}}(\mathbf{x}) + o_{\mathbb{P}}\big((nh^d)^{-1/2}\big),$$

because the reduced-form nonlinear remainders are $o_{\mathbb{P}}((nh^d)^{-1/2})$ by the pointwise analogue of Lemma SA-4, and the pointwise reduced-form rates give $\widehat{\vartheta}_A(\mathbf{x}) - \vartheta_A(\mathbf{x}) = O_{\mathbb{P}}(\rho_{n,\mathbf{x}})$ for $A \in \{Y, W\}$. Since $|\vartheta_W(\mathbf{x})| > 0$ and $\rho_{n,\mathbf{x}} = o(1)$, $\widehat{\vartheta}_W(\mathbf{x})$ is bounded away from zero with probability approaching one. Therefore the exact Wald remainder satisfies $\mathfrak{R}_{\mathtt{F}}(\mathbf{x}) = O_{\mathbb{P}}(\rho^2_{n,\mathbf{x}}) = o_{\mathbb{P}}((nh^d)^{-1/2})$ by the condition $\sqrt{nh^d}\rho^2_{n,\mathbf{x}} \to 0$. The separate small-bias condition makes $\mathfrak{B}_{\mathtt{F}}(\mathbf{x}) = o((nh^d)^{-1/2})$.

The leading term $\mathfrak{L}_{\mathrm{F}}(\mathbf{x})$ is a sum of independent mean-zero variables with variance $\Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}$. The moment assumptions and the bound $nh^d\Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \asymp 1$ verify Lyapunov's condition, giving

$$\frac{\mathfrak{L}_{\mathrm{F}}(\mathbf{x})}{\sqrt{\Xi_{\mathrm{F},\mathbf{x},\mathbf{x}}}} \rightsquigarrow \mathsf{Normal}(0,1).$$

The pointwise version of Lemma SA-8 gives $\widehat{\Xi}_{\mathrm{F},\mathbf{x},\mathbf{x}}/\Xi_{\mathrm{F},\mathbf{x},\mathbf{x}} \to_{\mathbb{P}} 1$. Slutsky's theorem therefore gives the normal approximation for $\widehat{\mathrm{T}}_{\mathrm{F}}(\mathbf{x})$ and the confidence interval coverage. □

SA-8.26 Proof of Theorem SA-18

Start from (SA-7). Lemma SA-4, applied to $A = Y$ and $A = W$, gives the uniform bound for the reduced-form nonlinear terms. Lemma SA-8 gives uniform consistency of the feasible standard error. The contribution of the reduced-form approximation biases is

$$\sqrt{nh^d}\sup_{\mathbf{x}\in\mathscr{B}}|\mathfrak{B}_{\mathrm{F}}(\mathbf{x})|.$$

Finally, (SA-8) gives the Wald-ratio contribution $\sqrt{nh^d}\rho_n^2$. Combining these bounds gives the displayed result. □

SA-8.27 Proof of Theorem SA-19

The proof is the fuzzy analogue of the proof of Theorem SA-9. The linearized fuzzy statistic can be written as an empirical process indexed by $\mathbf{x} \in \mathscr{B}$:

$$\overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f_{\mathrm{F},\mathbf{x}}(\mathbf{X}_i, Y_i, W_i),$$

where $f_{\mathrm{F},\mathbf{x}}$ is centered and equals a bounded linear combination, with coefficients $\mathfrak{w}(\mathbf{x})$, of the reduced-form distance-kernel terms for Y_i and W_i. The denominator condition $\inf_{\mathbf{x}\in\mathscr{B}}|\vartheta_W(\mathbf{x})| > 0$, the smooth reduced-form conditional means, and the metric equivalence in Assumption SA-2(ii) imply that $\mathfrak{w}(\mathbf{x})$ is bounded and Lipschitz on $\mathscr{B}$. The entropy calculation for Theorem SA-9 is therefore unchanged up to constants: adding the bounded treatment-receipt coordinate and bounded Lipschitz weights preserves the same VC-type covering bound. The envelope can be taken as $Ch^{-d/2}(|Y_i| + 1)$, because W_i is binary. The moment bounds follow from Assumption SA-1(v) and boundedness of W_i.

Applying Lemma SA-16 to $\{f_{\mathrm{F},\mathbf{x}} : \mathbf{x} \in \mathscr{B}\} \cup \{-f_{\mathrm{F},\mathbf{x}} : \mathbf{x} \in \mathscr{B}\}$ gives the displayed Gaussian approximation rate. The covariance of the constructed Gaussian process is, by construction, the covariance of $\overline{\mathrm{T}}_{\mathrm{F}}$. □

SA-8.28 Proof of Theorem SA-20

Theorem SA-18 and the stated rate conditions imply

$$\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}_{\mathrm{F}}(\mathbf{x}) - \overline{\mathrm{T}}_{\mathrm{F}}(\mathbf{x})| = o_{\mathbb{P}}\big((\log n)^{-1/2}\big).$$

Theorem SA-19 approximates the supremum of the infeasible linearized statistic by the supremum of a Gaussian process. Lemma SA-8, together with the same finite-net Gaussian comparison and anti-concentration

argument used in the proof of Theorem SA-10, replaces this infeasible Gaussian process by the feasible conditional process $\widehat{Z}_{\mathtt{F}}$. Thus,

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{\mathrm{T}}_{\mathtt{F}}(\mathbf{x})|\le u\right)-\mathbb{P}\left(\sup_{\mathbf{x}\in\mathscr{B}}|\widehat{Z}_{\mathtt{F}}(\mathbf{x})|\le u\mid\mathbf{V}\right)\right|=o_{\mathbb{P}}(1).$$

The coverage statement follows from the definition of $\widehat{\mathrm{I}}_{\alpha,\mathtt{F}}$ and the conditional quantile $\mathfrak{q}_\alpha$. □

SA-8.29 Proof of Lemma SA-9

By Fubini's theorem and the definition of $\Xi_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}$,

$$\begin{aligned}\mathbb{V}[\mathfrak{L}_{\mathtt{F},\mathtt{WBATE}}]&=\int_{\mathscr{B}}\int_{\mathscr{B}}\mathbb{C}\mathrm{ov}[\mathfrak{L}_{\mathtt{F}}(\mathbf{x}_1),\mathfrak{L}_{\mathtt{F}}(\mathbf{x}_2)]w(\mathbf{x}_1)w(\mathbf{x}_2)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_1)\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}_2)\\&=\Xi_{\mathtt{F},\mathtt{WBATE}}.\end{aligned}$$

The denominator condition implies that $\mathfrak{w}(\cdot)$ is uniformly bounded on $\mathscr{B}$. Compact support of K and equivalence of $\mathscr{d}$ and the Euclidean distance imply that $\Xi_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}=0$ unless $\|\mathbf{x}_1-\mathbf{x}_2\|\lesssim h$. Therefore, as in the proof of Lemma SA-6, the effective integration set has product boundary measure $O(h^{d-1})$ under (SA-4). The reduced-form covariance kernels are uniformly of order $(nh^d)^{-1}$, and hence

$$\Xi_{\mathtt{F},\mathtt{WBATE}}\lesssim\frac{h^{d-1}}{nh^d}=\frac{1}{nh}.$$

The matching lower bound follows from (SA-10).

For the plug-in variance, the proof of Lemma SA-8 gives the uniform covariance approximation

$$\sup_{\mathbf{x}_1,\mathbf{x}_2\in\mathscr{B}}nh^d\left|\widehat{\Xi}_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}-\Xi_{\mathtt{F},\mathbf{x}_1,\mathbf{x}_2}\right|=o_{\mathbb{P}}(1),$$

using only the denominator nondegeneracy and the stated rate conditions for the consistency of $\widehat{\mathfrak{w}}(\cdot)$. The same compact-support restriction applies to the feasible covariance kernel. Integrating the uniform error over the effective set of measure $O(h^{d-1})$ gives

$$\left|\widehat{\Xi}_{\mathtt{F},\mathtt{WBATE}}-\Xi_{\mathtt{F},\mathtt{WBATE}}\right|=o_{\mathbb{P}}\Big(\frac{h^{d-1}}{nh^d}\Big)=o_{\mathbb{P}}\Big(\frac{1}{nh}\Big).$$

The relative consistency statement follows after dividing by the lower bound in (SA-10). □

SA-8.30 Proof of Lemma SA-10

Because $\inf_{\mathbf{x}\in\mathscr{B}}|\vartheta_W(\mathbf{x})|>0$ and the reduced-form conditional means are continuous, the weight vector $\mathfrak{w}(\mathbf{x})$ is uniformly bounded over $\mathscr{B}$. The fuzzy nonlinear term can be written as a finite linear combination of reduced-form WBATE nonlinear terms:

$$\mathfrak{Q}_{\mathtt{F},\mathtt{WBATE}}=\int_{\mathscr{B}}\mathfrak{w}(\mathbf{x})^\top\big(\mathfrak{Q}_Y(\mathbf{x}),\mathfrak{Q}_W(\mathbf{x})\big)^\top w(\mathbf{x})\,\mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}).$$

The proof of Lemma SA-7 applies to each reduced-form component, with the bounded deterministic weights $w(\mathbf{x})\mathfrak{w}_Y(\mathbf{x})$ and $w(\mathbf{x})\mathfrak{w}_W(\mathbf{x})$ in place of $w(\mathbf{x})$. The moment bounds for terms involving Y follow from

Assumption SA-1(v), and those involving W are immediate from $W_i(t) \in \{0,1\}$. Hence

$$\mathfrak{Q}_{\mathtt{F},\mathtt{WBATE}} = O_{\mathbb{P}}\Big(\frac{1}{nh^d}\Big).$$

For the Wald-ratio remainder, (SA-8) and the integrability of w give

$$|\mathfrak{R}_{\mathtt{F},\mathtt{WBATE}}| \leq \int_{\mathscr{B}} |\mathfrak{R}_{\mathtt{F}}(\mathbf{x})| w(\mathbf{x})\, \mathrm{d}\mathfrak{H}^{d-1}(\mathbf{x}) \lesssim_{\mathbb{P}} \rho_n^2.$$

The final negligibility claim follows because

$$\frac{1/(nh^d)}{1/\sqrt{nh}} = \frac{1}{\sqrt{nh^{2d-1}}},$$

which is $o(1)$ under $nh^{2d-1} \to \infty$, while $\sqrt{nh}\rho_n^2 \to 0$ by assumption. □

SA-8.31 Proof of Theorem SA-21

The decomposition (SA-9), Lemma SA-9, and Lemma SA-10 give

$$\Big|\widehat{\xi}_{\mathtt{WBATE}} - \xi_{\mathtt{WBATE}}\Big| \leq |\mathfrak{B}_{\mathtt{F},\mathtt{WBATE}}| + O_{\mathbb{P}}\Big(\frac{1}{\sqrt{nh}}\Big) + O_{\mathbb{P}}\Big(\frac{1}{nh^d}\Big) + O_{\mathbb{P}}(\rho_n^2),$$

which is the stated rate. □

SA-8.32 Proof of Theorem SA-22

By (SA-9), Lemma SA-10, the small-bias condition, the condition $\sqrt{nh}\rho_n^2 \to 0$, and the aggregate nondegeneracy condition,

$$\frac{\widehat{\xi}_{\mathtt{WBATE}} - \xi_{\mathtt{WBATE}} - \mathfrak{L}_{\mathtt{F},\mathtt{WBATE}}}{\sqrt{\Xi_{\mathtt{F},\mathtt{WBATE}}}} = o_{\mathbb{P}}(1).$$

Lemma SA-9 gives $\widehat{\Xi}_{\mathtt{F},\mathtt{WBATE}}/\Xi_{\mathtt{F},\mathtt{WBATE}} \to_{\mathbb{P}} 1$.

Write $\mathfrak{L}_{\mathtt{F},\mathtt{WBATE}} = \sum_{i=1}^n \ell_{i,\mathtt{F}}$, where the $\ell_{i,\mathtt{F}}$ are the independent mean-zero summands obtained by integrating the reduced-form fuzzy linear influence functions over $\mathscr{B}$. By construction, $\sum_{i=1}^n \mathbb{E}[\ell_{i,\mathtt{F}}^2] = \Xi_{\mathtt{F},\mathtt{WBATE}}$. The same localization calculation used in the proof of Theorem SA-12, now with envelope proportional to $|Y_i|+1$ because W_i is binary and $\mathfrak{w}(\cdot)$ is bounded, gives

$$\sum_{i=1}^n \mathbb{E}\left[\left|\frac{\ell_{i,\mathtt{F}}}{\sqrt{\Xi_{\mathtt{F},\mathtt{WBATE}}}}\right|^3\right] \lesssim \frac{1}{\sqrt{nh}} \to 0.$$

Thus Lyapunov's central limit theorem yields

$$\frac{\mathfrak{L}_{\mathtt{F},\mathtt{WBATE}}}{\sqrt{\Xi_{\mathtt{F},\mathtt{WBATE}}}} \rightsquigarrow \mathsf{Normal}(0,1).$$

Slutsky's theorem gives the feasible statistic result, and the confidence interval coverage follows from the standard normal critical value. □

SA-8.33 Proof of Theorem SA-23

The supremum functional is one-Lipschitz under the sup-norm:

$$\left|\widehat{\xi}_{\texttt{LBATE}} - \xi_{\texttt{LBATE}}\right| \leq \sup_{\mathbf{x}\in\mathscr{B}} |\widehat{\xi}(\mathbf{x}) - \xi(\mathbf{x})|.$$

The result follows from the uniform part of Theorem SA-16. □

SA-8.34 Proof of Theorem SA-24

On the event

$$\xi(\mathbf{x}) \in \left[\widehat{\xi}(\mathbf{x}) - \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\texttt{F},\mathbf{x},\mathbf{x}}}, \widehat{\xi}(\mathbf{x}) + \mathfrak{q}_\alpha \sqrt{\widehat{\Xi}_{\texttt{F},\mathbf{x},\mathbf{x}}}\right] \quad \text{for all } \mathbf{x} \in \mathscr{B},$$

taking suprema over $\mathbf{x} \in \mathscr{B}$ implies

$$\xi_{\texttt{LBATE}} \in \widehat{\mathrm{I}}_{\alpha,\texttt{F},\texttt{LBATE}}.$$

Theorem SA-20 gives the first event probability $1 - \alpha + o(1)$, which proves the result. □

SA-8.35 Proof of Theorem 6

Let $0 < s < 1/6$ be a parameter that is chosen later. Consider a C^1 function $\Theta_s : [0,1] \to [\underline{\theta}, \pi]$, for some fixed $\underline{\theta} > 0$, satisfying

$$\Theta_s(r) = \pi \quad (0 \leq r \leq 2s), \qquad \frac{\sin(\Theta_s(r))}{\Theta_s(r)} = \frac{2}{\pi} - \frac{s}{r} \quad (3s \leq r \leq 1), \qquad \sup_{0<r\leq 1} r|\Theta_s'(r)| \leq C,$$

with a smooth interpolation on $[2s, 3s]$. Such a function exists because $\theta \mapsto \sin(\theta)/\theta$ maps $(0, \pi]$ continuously and monotonically onto $[0, 1)$.

Data Generating Process $\mathbb{P}_0$.

Let

$$\mathscr{X}_0 = \{r(\cos\theta, \sin\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \Theta_s(r)\}.$$

Suppose $\mathbf{X}_i$ has Lebesgue density

$$f_{0,X}(r(\cos\theta, \sin\theta)) = \frac{2}{\Theta_s(r)}, \qquad 0 \leq r \leq 1, \quad 0 \leq \theta \leq \Theta_s(r),$$

and define

$$\mu_0(x_1, x_2) = \frac{1}{2} + \frac{x_1}{100}, \qquad (x_1, x_2) \in \mathbb{R}^2.$$

Let $Y_i = \mathbb{1}(\varepsilon_i \leq \mu_0(\mathbf{X}_i))$, where $(\varepsilon_i : i = 1, \ldots, n)$ are i.i.d. $\mathsf{Uniform}([0,1])$ random variables independent of $(\mathbf{X}_i : i = 1, \ldots, n)$. Let $\eta_0(r) = \mathbb{E}_{\mathbb{P}_0}[Y_i | \, \|\mathbf{X}_i\| = r]$. The boundary $\mathrm{bd}(\mathscr{X}_0)$ is rectifiable with uniformly bounded length.

Data Generating Process $\mathbb{P}_1$.

Let

$$\mathcal{X}_1 = \{r(\cos\theta, \sin\theta) : 0 \le r \le 1,\ 0 \le \theta \le \pi/2\},$$

let $\mathbf{X}_i$ have density $f_{1,X} = 4/\pi$ on $\mathcal{X}_1$, and define

$$\mu_1(x_1, x_2) = \frac{1}{2} + \frac{x_1 - s}{100}, \qquad (x_1, x_2) \in \mathbb{R}^2.$$

Let $Y_i = \mathbb{1}(\varepsilon_i \le \mu_1(\mathbf{X}_i))$ with the same independent uniform construction as above. Let $\eta_1(r) = \mathbb{E}_{\mathbb{P}_1}[Y_i \,|\, \|\mathbf{X}_i\| = r]$. The boundary $\mathrm{bd}(\mathcal{X}_1)$ is rectifiable.

Figure SA-2 displays the two support sets used in the minimax lower-bound construction.

Minimax Lower Bound.

First, under both models, $\mathbb{P}_j(\|\mathbf{X}_i\| \le r) = r^2$ for all $0 \le r \le 1$:

$$\mathbb{P}_0(\|\mathbf{X}_i\| \le r) = \int_0^r \int_0^{\Theta_s(\rho)} \frac{2}{\Theta_s(\rho)} \rho \,\mathrm{d}\theta \,\mathrm{d}\rho = r^2 = \int_0^r \int_0^{\pi/2} \frac{4}{\pi} \rho \,\mathrm{d}\theta \,\mathrm{d}\rho = \mathbb{P}_1(\|\mathbf{X}_i\| \le r).$$

The two data-generating processes belong to $\mathscr{P}_{\mathtt{NP}}$ for a fixed sufficiently large L: the supports are compact subsets of $[-L, L]^2$ with rectifiable boundaries of uniformly bounded length, the densities are continuous and uniformly bounded above and below on their supports, and the Bernoulli conditional means and variances satisfy the required smoothness and boundedness conditions. Hence, choosing $(0, 0)$ as the point of evaluation in both $\mathbb{P}_0$ and $\mathbb{P}_1$, the one-observation KL distance for the distance-based experiment based on $(\|\mathbf{X}_i\|, Y_i)$ is

$$d_{\mathrm{KL}}(\mathbb{P}_0(\|\mathbf{X}_i\|, Y_i), \mathbb{P}_1(\|\mathbf{X}_i\|, Y_i)) = 2\int_0^1 d_{\mathrm{KL}}(\mathsf{Bernoulli}(\eta_0(r)), \mathsf{Bernoulli}(\eta_1(r)))r \,\mathrm{d}r.$$

Under $\mathbb{P}_0$ and $\mathbb{P}_1$,

$$\eta_0(r) = \frac{1}{2} + \frac{r}{100}\frac{\sin(\Theta_s(r))}{\Theta_s(r)}, \qquad \eta_1(r) = \frac{1}{2} + \frac{1}{100}\Big(\frac{2r}{\pi} - s\Big).$$

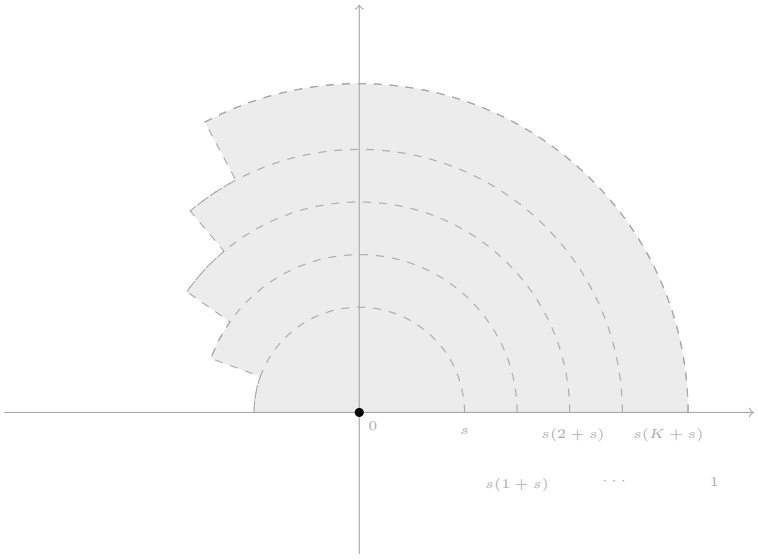
Thus $\eta_0(r) = \eta_1(r)$ for all $r \in [3s, 1]$. For $r \in [0, 3s]$, both probabilities are bounded away from zero and one, and $|\eta_0(r) - \eta_1(r)| \le Cs$. Therefore,

$$\int_0^1 d_{\mathrm{KL}}(\mathsf{Bernoulli}(\eta_0(r)), \mathsf{Bernoulli}(\eta_1(r)))r \,\mathrm{d}r \le C\int_0^{3s} (\eta_0(r) - \eta_1(r))^2 r \,\mathrm{d}r \le Cs^4.$$

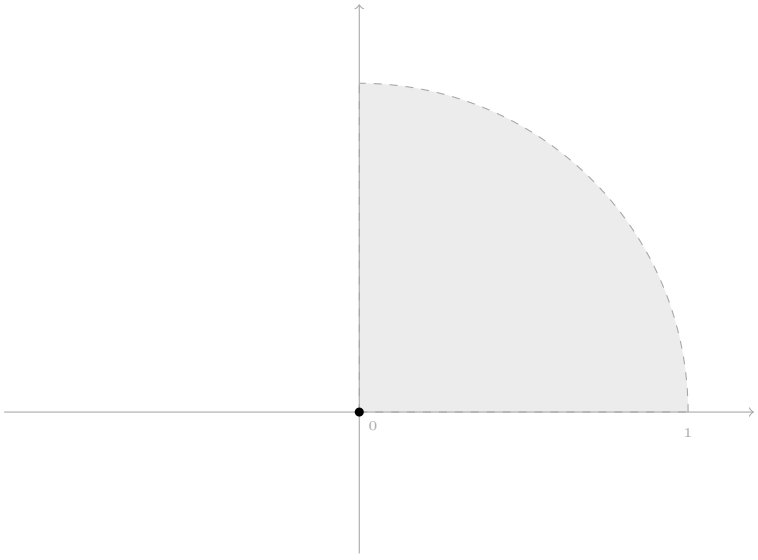
Moreover, $|\mu_0(0,0) - \mu_1(0,0)| = s/100$. Hence, by Tsybakov [2008, Theorem 2.2 (iii)], take $Cs_*^4 = \log(2)/n$, and conclude that the pointwise risk at $\mathbf{x} = 0$ is bounded below by a constant multiple of s_*. Since the uniform loss dominates the pointwise loss at any boundary point,

$$\inf_{T_n \in \mathscr{T}} \sup_{\mathbb{P} \in \mathscr{P}_{\mathtt{NP}}} \mathbb{E}_{\mathbb{P}}\Big[\sup_{\mathbf{x} \in \mathscr{B}(\mathbb{P})} |T_n(\mathbf{U}_n(\mathbf{x})) - \mu(\mathbf{x})|\Big] \ge \frac{1}{1600} s_* \gtrsim n^{-\frac{1}{4}}.$$

This concludes the proof. □



$\mathscr{X}_0$ from DGP $\mathbb{P}_0$



$\mathscr{X}_1$ from DGP $\mathbb{P}_1$

Figure SA-2: Support sets in the two-point minimax lower-bound construction.

SA-9 Simulation Study

The score $\mathbf{X}_i = (X_{1i}, X_{2i})^\top$ is generated as $(100B_{1i} - 25, 100B_{2i} - 25)^\top$, where B_{1i} and B_{2i} are independent $\mathsf{Beta}(3,4)$ random variables, a choice that approximately reproduces salient features of the SPP dataset. The treatment assignment region is the same L-shaped region used in the empirical application,

$$\mathscr{A}_1 = \{\mathbf{x} = (x_1, x_2)^\top : x_1 \geq 0,\ x_2 \geq 0\}, \qquad \mathscr{A}_0 = \mathscr{X} \setminus \mathscr{A}_1,$$

with boundary $\mathscr{B}$ given by the two coordinate-axis segments and a kink at $\mathbf{x} = (0,0)^\top$. The simulation uses 21 evaluation points $\{\mathbf{b}_1, \ldots, \mathbf{b}_{21}\} \subset \mathscr{B}$, with $\mathbf{b}_{11} = (0,0)^\top$ corresponding to the kink.

The simulation design sets $n = 20,000$ and uses $5,000$ Monte Carlo replications. We evaluate performance in terms of point estimation and uncertainty quantification, both pointwise and uniformly over the evaluation grid. We consider two related Monte Carlo exercises. The first is a sharp-design exercise, which uses only the reduced-form or intention-to-treat (ITT) outcome equation and therefore reports sharp distance-based results for the BATEC. The second is a fuzzy-design exercise, which jointly generates the reduced-form outcome and treatment-receipt status so that the ITT, first-stage (FS), and fuzzy Wald analyses can all be performed on the same simulated samples.

Throughout the simulation study, the distance-based estimator uses a local linear polynomial, $p = 1$. The DGP varies the polynomial order s of the underlying bivariate conditional mean surfaces. We consider $s \in \{1, 2\}$, corresponding to linear and quadratic specifications. In addition, each mean specification is paired with either homoskedastic or heteroskedastic outcome disturbances.

SA-9.1 Data-Generating Process

For each Monte Carlo replication, draw $\mathbf{X}_i = (X_{1i}, X_{2i})^\top$ independently from the distribution described above and set $T_i = \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_1)$. Let $\mathbf{m}_s(\cdot)$ be a monomial vector of degree s, that is,

$$\mathbf{m}_0(\mathbf{x}) = 1, \qquad \mathbf{m}_1(\mathbf{x}) = (1, x_1, x_2)^\top, \qquad \text{and} \qquad \mathbf{m}_2(\mathbf{x}) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)^\top.$$

We consider DGP polynomial specifications of order $s \in \{1, 2\}$, using the monomial vector evaluated at the unscaled score.

Sharp Design: ITT Equation.

The sharp-design simulation uses an outcome-only DGP. For each $t \in \{0, 1\}$ and $i = 1, \ldots, n$, define

$$\mu_{Y,t}(\mathbf{X}_i) = \mathbf{m}_s(\mathbf{X}_i)^\top \boldsymbol{\beta}_{Y,t,s}, \qquad \sigma^2_{Y,t,v}(\mathbf{X}_i) = \exp\big(\mathbf{m}_v(\mathbf{X}_i)^\top \boldsymbol{\delta}_{Y,t,v}\big),$$

where $v \in \{0, 2\}$ indexes the homoskedastic and heteroskedastic specifications. Draw $u_{Y,t,i} \sim \mathsf{Normal}(0,1)$ independently across i and t and independently of all score variables, and set

$$\varepsilon_{Y,t,i} = \sigma_{Y,t,v}(\mathbf{X}_i) u_{Y,t,i}.$$

Then

$$Y_i(t) = \mu_{Y,t}(\mathbf{X}_i) + \varepsilon_{Y,t,i}.$$

The observed sharp-design outcome is

$$Y_i = Y_i(0)(1 - T_i) + Y_i(1)T_i.$$

The target at each boundary point $\mathbf{b} \in \mathscr{B}$ is

$$\tau(\mathbf{b}) = \mathbb{E}[Y_i(1) - Y_i(0) \mid \mathbf{X}_i = \mathbf{b}] = \mu_{Y,1}(\mathbf{b}) - \mu_{Y,0}(\mathbf{b}).$$

This is the design presented in the paper, where we suppress the outcome subscript and write μ_t, σ_t, and $u_{t,i}$ in place of $\mu_{Y,t}$, $\sigma_{Y,t,v}$, and $u_{Y,t,i}$ for notation simplicity.

Fuzzy Design: ITT, FS, and Wald Equations.

For each $t \in \{0, 1\}$ and $i = 1, \ldots, n$, define

$$\begin{aligned}
\mu_{Y,t}(\mathbf{X}_i) &= \mathbf{m}_s(\mathbf{X}_i)^\top \boldsymbol{\beta}_{Y,t,s}, \\
\mu_{W,t}(\mathbf{X}_i) &= \Lambda(\eta_{W,t}(\mathbf{X}_i)), \qquad \eta_{W,t}(\mathbf{X}_i) = \mathbf{m}_s(\mathbf{X}_i)^\top \boldsymbol{\beta}_{W,t,s},
\end{aligned}$$

where $\Lambda(a) = 1/(1 + \exp(-a))$. We also consider homoskedastic and heteroskedastic specifications for the residualized outcome innovation. Let $v \in \{0, 2\}$ index the variance specification, where $v = 0$ is homoskedastic and $v = 2$ is heteroskedastic:

$$\sigma^2_{Y,t,v}(\mathbf{x}) = \exp\left(\mathbf{m}_v(\mathbf{x})^\top \boldsymbol{\delta}_{Y,t,v}\right).$$

Given these conditional mean and variance functions, draw $U_{W,t,i} \sim \mathsf{Uniform}(0, 1)$ and $u_{Y,t,i} \sim \mathsf{Normal}(0, 1)$ mutually independently across i and t and independently of all score variables, and set

$$\varepsilon_{Y,t,i} = \sigma_{Y,t,v}(\mathbf{X}_i) u_{Y,t,i}.$$

The potential treatment statuses and treatment-indexed potential outcomes are

$$\begin{aligned}
W_i(t) &= \mathbb{1}\{U_{W,t,i} \le \mu_{W,t}(\mathbf{X}_i)\}, \\
Y_i(t, w) &= \mu_{Y,t}(\mathbf{X}_i) + \lambda_t\{w - \mu_{W,t}(\mathbf{X}_i)\} + \varepsilon_{Y,t,i}, \qquad w \in \{0, 1\}.
\end{aligned}$$

The reduced-form potential outcome induced by assignment t is $Y_i(t, W_i(t))$. Equivalently,

$$Y_i(t, W_i(t)) = \mu_{Y,t}(\mathbf{X}_i) + \lambda_t\{W_i(t) - \mu_{W,t}(\mathbf{X}_i)\} + \varepsilon_{Y,t,i}.$$

The observed outcomes and treatment status are

$$\begin{aligned}
Y_i &= Y_i(0, W_i(0))(1 - T_i) + Y_i(1, W_i(1))T_i, \\
W_i &= W_i(0)(1 - T_i) + W_i(1)T_i, \qquad T_i = \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_1),
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

In the reduced-form notation used for sharp estimation and inference, $Y_i(t)$ is shorthand for $Y_i(t, W_i(t))$. This centered specification preserves the reduced-form outcome surface because $\mathbb{E}[Y_i(t, W_i(t))|\mathbf{X}_i = \mathbf{x}] =$

$\mu_{Y,t}(\mathbf{x})$, while allowing the potential outcome and potential treatment status equations to be correlated:

$$\mathbb{E}[W_i(t)|\mathbf{X}_i = \mathbf{x}] = \mu_{W,t}(\mathbf{x}),$$
$$\mathbb{C}\text{ov}(Y_i(t, W_i(t)), W_i(t)|\mathbf{X}_i = \mathbf{x}) = \lambda_t \mu_{W,t}(\mathbf{x})\{1 - \mu_{W,t}(\mathbf{x})\}.$$

The heteroskedastic specification changes the conditional variance of the residualized outcome innovation $\varepsilon_{Y,t,i}$, but it does not change the conditional mean surfaces or the first-stage equation. In the fuzzy design, the conditional variance of the reduced-form outcome also includes the treatment-receipt component induced by $\lambda_t\{W_i(t) - \mu_{W,t}(\mathbf{X}_i)\}$. Consequently, for each boundary point $\mathbf{b} \in \mathscr{B}$,

$$\begin{aligned}
\tau_W(\mathbf{b}) &= \mathbb{E}[W_i(1) - W_i(0)|\mathbf{X}_i = \mathbf{b}] = \mu_{W,1}(\mathbf{b}) - \mu_{W,0}(\mathbf{b}), \\
\tau_Y(\mathbf{b}) &= \mathbb{E}[Y_i(1, W_i(1)) - Y_i(0, W_i(0))|\mathbf{X}_i = \mathbf{b}] = \mu_{Y,1}(\mathbf{b}) - \mu_{Y,0}(\mathbf{b}) = \mathbf{m}_s(\mathbf{b})^\top \left(\boldsymbol{\beta}_{Y,1,s} - \boldsymbol{\beta}_{Y,0,s}\right), \\
\zeta(\mathbf{b}) &= \frac{\tau_Y(\mathbf{b})}{\tau_W(\mathbf{b})}.
\end{aligned}$$

Within each fuzzy-design Monte Carlo replication, the sharp estimator applied to Y_i targets the reduced-form or ITT curve $\tau_Y(\mathbf{b})$, the same estimator applied to W_i targets the first-stage curve $\tau_W(\mathbf{b})$, and the fuzzy estimator targets the Wald curve $\zeta(\mathbf{b})$. Thus, the ITT, first-stage, and fuzzy simulation results are different summaries of the same simulated samples rather than results from separate fuzzy DGPs. The fuzzy design therefore extends the sharp ITT equation by adding a treatment-receipt equation and allowing the reduced-form outcome and treatment-receipt shocks to be correlated through λ_t.

SA-9.2 Calibration from the SPP Data

The DGP parameters are calibrated once from the SPP data and then held fixed across Monte Carlo replications. We use the following steps separately for each DGP polynomial order $s \in \{1, 2\}$. The conditional mean calibration follows the JASA simulation design; the additional calibration determines the homoskedastic or heteroskedastic variance of the innovation entering the outcome equation. The sharp-design exercise uses the calibrated ITT objects $\boldsymbol{\beta}_{Y,t,s}$ and $\boldsymbol{\delta}_{Y,t,v}$; the fuzzy-design exercise additionally uses the treatment-receipt coefficients $\boldsymbol{\beta}_{W,t,s}$ and the residual-dependence parameters λ_t.

Step 1. Fix the DGP polynomial order $s \in \{1, 2\}$ and define the side-specific calibration samples $\mathscr{I}_t = \{i : T_i = t\}$ and $N_t = |\mathscr{I}_t|$, for $t \in \{0, 1\}$.

Step 2. For each side t, estimate the outcome regression coefficients by side-specific least squares:

$$\widehat{\boldsymbol{\beta}}_{Y,t,s} = \arg\min_{\boldsymbol{\beta}} \sum_{i \in \mathscr{I}_t} \{Y_i - \mathbf{m}_s(\mathbf{X}_i)^\top \boldsymbol{\beta}\}^2.$$

This gives the fitted reduced-form outcome surface $\widehat{\mu}_{Y,t}(\mathbf{x}) = \mathbf{m}_s(\mathbf{x})^\top \widehat{\boldsymbol{\beta}}_{Y,t,s}$.

Step 3. For each side t, estimate the treatment-status regression coefficients by side-specific logit:

$$\widehat{\boldsymbol{\beta}}_{W,t,s} = \arg\max_{\boldsymbol{\beta}} \sum_{i \in \mathscr{I}_t} \left[W_i \mathbf{m}_s(\mathbf{X}_i)^\top \boldsymbol{\beta} - \log\{1 + \exp(\mathbf{m}_s(\mathbf{X}_i)^\top \boldsymbol{\beta})\}\right].$$

This gives the fitted first-stage surface $\widehat{\mu}_{W,t}(\mathbf{x}) = \Lambda(\mathbf{m}_s(\mathbf{x})^\top \widehat{\boldsymbol{\beta}}_{W,t,s})$.

Step 4. Construct side-specific residuals in the calibration sample:

$$\widehat{\varepsilon}_{Y,i} = Y_i - \widehat{\mu}_{Y,T_i}(\mathbf{X}_i), \qquad \widehat{\varepsilon}_{W,i} = W_i - \widehat{\mu}_{W,T_i}(\mathbf{X}_i).$$

Step 5. Estimate the residual dependence parameter on each side:

$$\widehat{\lambda}_t = \frac{\sum_{i\in\mathscr{I}_t} \widehat{\varepsilon}_{Y,i}\widehat{\varepsilon}_{W,i}}{\sum_{i\in\mathscr{I}_t} \widehat{\varepsilon}^2_{W,i}}, \qquad t \in \{0,1\}.$$

Define the estimated residualized outcome innovation, distinct from the standardized simulation draw $u_{Y,t,i}$,

$$\widehat{\omega}_{Y,i} = \widehat{\varepsilon}_{Y,i} - \widehat{\lambda}_{T_i}\widehat{\varepsilon}_{W,i}.$$

For the homoskedastic specification, estimate the intercept-only log-variance coefficient

$$\widehat{\boldsymbol{\delta}}_{Y,t,0} = \log\left(\frac{1}{N_t}\sum_{i\in\mathscr{I}_t} \widehat{\omega}^2_{Y,i}\right), \qquad t \in \{0,1\}.$$

For the heteroskedastic specification, estimate the side-specific log-variance regression for the residualized outcome innovation:

$$\widehat{\boldsymbol{\delta}}_{Y,t,2} = \arg\min_{\boldsymbol{\delta}} \sum_{i\in\mathscr{I}_t} \left[\log\{\max(\widehat{\omega}^2_{Y,i}, \epsilon_0)\} - \mathbf{m}_2(\mathbf{X}_i)^\top\boldsymbol{\delta}\right]^2,$$

where $\epsilon_0 > 0$ is a small truncation constant used only to avoid taking logs of zero.

Step 6. Set the Monte Carlo DGP parameters equal to the calibration estimates: $\boldsymbol{\beta}_{Y,t,s} = \widehat{\boldsymbol{\beta}}_{Y,t,s}$, $\boldsymbol{\beta}_{W,t,s} = \widehat{\boldsymbol{\beta}}_{W,t,s}$, $\lambda_t = \widehat{\lambda}_t$, $\boldsymbol{\delta}_{Y,t,0} = \widehat{\boldsymbol{\delta}}_{Y,t,0}$, and $\boldsymbol{\delta}_{Y,t,2} = \widehat{\boldsymbol{\delta}}_{Y,t,2}$ for $t \in \{0,1\}$.

Step 7. Before running the Monte Carlo replications, compute the true target functions analytically from the calibrated conditional mean surfaces. For each evaluation point $\mathbf{b} \in \mathscr{B}$, the sharp-design BATEC is

$$\tau(\mathbf{b}) = \mathbf{m}_s(\mathbf{b})^\top(\boldsymbol{\beta}_{Y,1,s} - \boldsymbol{\beta}_{Y,0,s}).$$

For the fuzzy-design exercise, the reduced-form/ITT, first-stage, and Wald targets are

$$\tau_Y(\mathbf{b}) = \mathbf{m}_s(\mathbf{b})^\top(\boldsymbol{\beta}_{Y,1,s} - \boldsymbol{\beta}_{Y,0,s}), \qquad \tau_W(\mathbf{b}) = \Lambda(\mathbf{m}_s(\mathbf{b})^\top\boldsymbol{\beta}_{W,1,s}) - \Lambda(\mathbf{m}_s(\mathbf{b})^\top\boldsymbol{\beta}_{W,0,s}),$$

and $\zeta(\mathbf{b}) = \frac{\tau_Y(\mathbf{b})}{\tau_W(\mathbf{b})}$. The same calibrated outcome surface is used in both exercises, so $\tau(\mathbf{b})$ and $\tau_Y(\mathbf{b})$ have the same algebraic form under this calibration; they are kept notationally distinct because they refer to different Monte Carlo exercises and different causal objects.

This calibration reproduces the empirical residual covariance between the outcome and take-up equations on each side of the boundary, while keeping the marginal reduced-form and first-stage surfaces fixed by the fitted SPP models. The homoskedastic and heteroskedastic DGPs share the same target curves for a given mean specification; they differ only in the conditional variance of the residualized outcome innovation.

SA-9.3 Monte Carlo Implementation

The following steps describe how to simulate from the calibrated model. They are repeated separately for each pair $(s, v) \in \{1, 2\} \times \{0, 2\}$. For the sharp-design exercise, the same score draws, assignment rule, evaluation grid, polynomial order, and bandwidth choices are used, but Steps 4–6 are simplified by keeping only the outcome equation Y_i and dropping $W_i(t)$, $\mu_{W,t}$, λ_t, and the Wald ratio.

Step 1. Fix the Monte Carlo inputs: the sample size n, the number of replications M, the evaluation grid $\{\mathbf{b}_1, \ldots, \mathbf{b}_{21}\}$ with $\mathbf{b}_{11}$ at the kink, the assignment regions $\mathscr{A}_0$ and $\mathscr{A}_1$, the local polynomial order $p = 1$, and the calibrated parameters. The sharp-design exercise uses only $\{\boldsymbol{\beta}_{Y,t,s}, \boldsymbol{\delta}_{Y,t,0}, \boldsymbol{\delta}_{Y,t,2} : t \in \{0, 1\}\}$, while the fuzzy-design exercise uses the full set

$$\{\boldsymbol{\beta}_{Y,t,s}, \boldsymbol{\beta}_{W,t,s}, \lambda_t, \boldsymbol{\delta}_{Y,t,0}, \boldsymbol{\delta}_{Y,t,2} : t \in \{0, 1\}\}.$$

Step 2. Before entering the Monte Carlo loop, evaluate the true target functions on the grid. For the sharp-design exercise this is only the BATEC target $\tau(\mathbf{b}_j)$. For the fuzzy-design exercise the targets are

$$\tau_Y(\mathbf{b}_j), \qquad \tau_W(\mathbf{b}_j), \qquad \zeta(\mathbf{b}_j), \qquad j = 1, \ldots, 21.$$

These target values are fixed across Monte Carlo replications and are common to the homoskedastic and heteroskedastic versions of a given mean specification.

Step 3. For replication $m = 1, \ldots, M$, draw independent score vectors $\mathbf{X}_i = (100B_{1i} - 25, 100B_{2i} - 25)^\top$, with B_{1i} and B_{2i} independent $\mathsf{Beta}(3, 4)$ random variables, and set $T_i = \mathbb{1}(\mathbf{X}_i \in \mathscr{A}_1)$.

Step 4. Form the design vectors $\mathbf{m}_s(\mathbf{X}_i)$ and compute the side-specific conditional outcome means and variances

$$\mu_{Y,t}(\mathbf{X}_i), \qquad \sigma^2_{Y,t,v}(\mathbf{X}_i), \qquad t \in \{0, 1\}.$$

Under homoskedasticity, $\sigma^2_{Y,t,v}(\mathbf{X}_i)$ is constant within side t. Under heteroskedasticity, it changes with $\mathbf{X}_i$ through the fitted log-variance model. For the fuzzy-design exercise, also compute $\mu_{W,t}(\mathbf{X}_i)$.

Step 5. For the sharp-design exercise, draw $u_{Y,t,i} \sim \mathsf{Normal}(0, 1)$ independently across i and t and independently of all score variables, and set

$$\varepsilon_{Y,t,i} = \sigma_{Y,t,v}(\mathbf{X}_i) u_{Y,t,i}.$$

Then construct $Y_i(t) = \mu_{Y,t}(\mathbf{X}_i) + \varepsilon_{Y,t,i}$. For the fuzzy-design exercise, draw $U_{W,t,i} \sim \mathsf{Uniform}(0, 1)$ and $u_{Y,t,i} \sim \mathsf{Normal}(0, 1)$ mutually independently across i and t and independently of all score variables, define $\varepsilon_{Y,t,i}$ as above, and construct

$$\begin{aligned} W_i(t) &= \mathbb{1}\{U_{W,t,i} \le \mu_{W,t}(\mathbf{X}_i)\}, \\ Y_i(t, W_i(t)) &= \mu_{Y,t}(\mathbf{X}_i) + \lambda_t\{W_i(t) - \mu_{W,t}(\mathbf{X}_i)\} + \varepsilon_{Y,t,i}. \end{aligned}$$

Step 6. Select observed variables using the assignment indicator. For the sharp-design exercise, $Y_i =$

$Y_i(0)(1-T_i)+Y_i(1)T_i$. For the fuzzy-design exercise,

$$W_i = W_i(0)(1-T_i)+W_i(1)T_i,$$
$$Y_i = Y_i(0,W_i(0))(1-T_i)+Y_i(1,W_i(1))T_i.$$

The simulated dataset for replication m is therefore $\{Y_i, T_i, \mathbf{X}_i : i=1,\dots,n\}$ in the sharp-design exercise and $\{Y_i, W_i, T_i, \mathbf{X}_i : i=1,\dots,n\}$ in the fuzzy-design exercise.

Step 7. For each evaluation point $\mathbf{b}_j$, compute the signed distance score $D_i(\mathbf{b}_j)$ using the Euclidean distance and the assignment side. Estimate the target curves using the distance-based local linear estimator.

Step 8. **Sharp/ITT Analysis**. In the sharp-design exercise, apply the sharp distance-based estimator to Y_i for the target $\tau(\mathbf{b}_j)$. In the fuzzy-design exercise, apply the same sharp estimator to the reduced-form outcome Y_i for the ITT target $\tau_Y(\mathbf{b}_j)$.

Step 9. **FS Analysis**. In the fuzzy-design exercise, apply the sharp distance-based estimator to W_i for the target $\tau_W(\mathbf{b}_j)$.

Step 10. **Fuzzy Analysis**. In the fuzzy-design exercise, apply the fuzzy distance-based estimator using Y_i as the outcome and W_i as the treatment-receipt status for the Wald target $\zeta(\mathbf{b}_j)$.

Step 11. Repeat the analysis using the bandwidth choices considered in the main text: the smooth-boundary selector, the kink-adaptive selector using the known kink location, the unknown-kink bandwidth rule, and the bandwidth induced by the univariate `rdrobust` selector.

Step 12. After all M replications, compute Monte Carlo summaries by comparing the stored estimates and confidence sets with the fixed analytic targets. For each target and each grid point, report bias, standard deviation, root mean squared error, pointwise coverage, and average interval length. For uniform inference, report simultaneous coverage and average band length over the grid. Finally, also recover WBATE and LBATE summaries and report their associated performance.

We use the sharp simulations to study the finite-sample behavior of the sharp BATEC estimator for $\tau(\mathbf{b})$. The fuzzy simulations additionally study the reduced-form/ITT estimators, the first-stage estimators, and the fuzzy Wald estimator. A complier-average causal interpretation of $\zeta(\mathbf{b})$ would require additional assumptions that are not imposed in the simulations. In particular, one would need a local exclusion restriction ruling out a direct effect of assignment on the outcome except through treatment take-up, a monotonicity condition ruling out defiers near the boundary, and a nonzero first-stage effect. Under those conditions, the Wald curve $\zeta(\mathbf{b})$ can be interpreted as a local average treatment effect for compliers at the boundary point $\mathbf{b}$.

SA-9.4 Results

The following tables report Monte Carlo summaries separately for the sharp estimator and, in the fuzzy-design simulations, for the reduced-form/ITT, first-stage, and fuzzy estimators. Results are reported separately by DGP polynomial order $s \in \{1,2\}$ and by homoskedastic or heteroskedastic outcome disturbances. We first report the sharp-design results. We then report two sets of fuzzy-design results: one using the ITT bandwidth selector, matching the main empirical specification, and one using the fuzzy bandwidth selector.

The sharp-design simulation results are reported first. Specifically:

- Tables SA-1–SA-4 report the sharp simulation results for the linear homoskedastic specification.
- Tables SA-5–SA-8 report the sharp simulation results for the linear heteroskedastic specification.
- Tables SA-9–SA-12 report the sharp simulation results for the quadratic homoskedastic specification.
- Tables SA-13–SA-16 report the sharp simulation results for the quadratic heteroskedastic specification.

The first set of fuzzy-design simulation results uses the ITT bandwidth selector. Specifically:

- Tables SA-17–SA-20 report the reduced-form (ITT) simulation results for the linear homoskedastic specification.
- Tables SA-21–SA-24 report the reduced-form (ITT) simulation results for the linear heteroskedastic specification.
- Tables SA-25–SA-28 report the reduced-form (ITT) simulation results for the quadratic homoskedastic specification.
- Tables SA-29–SA-32 report the reduced-form (ITT) simulation results for the quadratic heteroskedastic specification.
- Tables SA-33–SA-36 report the take-up (FS) simulation results for the linear homoskedastic specification.
- Tables SA-37–SA-40 report the take-up (FS) simulation results for the linear heteroskedastic specification.
- Tables SA-41–SA-44 report the take-up (FS) simulation results for the quadratic homoskedastic specification.
- Tables SA-45–SA-48 report the take-up (FS) simulation results for the quadratic heteroskedastic specification.
- Tables SA-49–SA-52 report the fuzzy simulation results for the linear homoskedastic specification.
- Tables SA-53–SA-56 report the fuzzy simulation results for the linear heteroskedastic specification.
- Tables SA-57–SA-60 report the fuzzy simulation results for the quadratic homoskedastic specification.
- Tables SA-61–SA-64 report the fuzzy simulation results for the quadratic heteroskedastic specification.

The second set of fuzzy-design simulation results uses the fuzzy bandwidth selector. Specifically:

- Tables SA-65–SA-68 report the reduced-form (ITT) simulation results for the linear homoskedastic specification.
- Tables SA-69–SA-72 report the reduced-form (ITT) simulation results for the linear heteroskedastic specification.
- Tables SA-73–SA-76 report the reduced-form (ITT) simulation results for the quadratic homoskedastic specification.

- Tables SA-77–SA-80 report the reduced-form (ITT) simulation results for the quadratic heteroskedastic specification.
- Tables SA-81–SA-84 report the take-up (FS) simulation results for the linear homoskedastic specification.
- Tables SA-85–SA-88 report the take-up (FS) simulation results for the linear heteroskedastic specification.
- Tables SA-89–SA-92 report the take-up (FS) simulation results for the quadratic homoskedastic specification.
- Tables SA-93–SA-96 report the take-up (FS) simulation results for the quadratic heteroskedastic specification.
- Tables SA-97–SA-100 report the fuzzy simulation results for the linear homoskedastic specification.
- Tables SA-101–SA-104 report the fuzzy simulation results for the linear heteroskedastic specification.
- Tables SA-105–SA-108 report the fuzzy simulation results for the quadratic homoskedastic specification.
- Tables SA-109–SA-112 report the fuzzy simulation results for the quadratic heteroskedastic specification.

The columns report the average bandwidths, average bias, standard deviation, root mean squared error, empirical coverage, and average interval or band length. For the Uniform row, empirical coverage and average length refer to uniform coverage and average uniform band length.

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.531	19.531	2120	4259	-0.001	0.033	0.033	0.958	0.240
$\mathbf{b}_2$	19.133	19.133	2135	4107	-0.001	0.034	0.034	0.956	0.243
$\mathbf{b}_3$	18.879	18.879	2182	3953	-0.000	0.035	0.035	0.955	0.245
$\mathbf{b}_4$	18.784	18.784	2267	3798	0.002	0.036	0.036	0.952	0.246
$\mathbf{b}_5$	18.826	18.826	2383	3637	0.004	0.035	0.036	0.947	0.246
$\mathbf{b}_6$	18.978	18.978	2512	3469	0.007	0.035	0.035	0.953	0.245
$\mathbf{b}_7$	19.090	19.090	2617	3257	0.008	0.034	0.035	0.960	0.247
$\mathbf{b}_8$	19.093	19.093	2675	2989	0.008	0.035	0.036	0.959	0.252
$\mathbf{b}_9$	19.128	19.128	2711	2714	0.006	0.036	0.037	0.953	0.258
$\mathbf{b}_{10}$	19.376	19.376	2758	2482	0.003	0.038	0.038	0.955	0.268
$\mathbf{b}_{11}$	21.264	21.264	3050	2652	0.000	0.038	0.038	0.951	0.279
$\mathbf{b}_{12}$	19.240	19.240	2736	2567	-0.006	0.037	0.038	0.955	0.263
$\mathbf{b}_{13}$	19.204	19.204	2705	2964	-0.012	0.036	0.038	0.954	0.252
$\mathbf{b}_{14}$	19.259	19.259	2637	3357	-0.014	0.035	0.038	0.951	0.245
$\mathbf{b}_{15}$	19.026	19.026	2464	3617	-0.011	0.035	0.037	0.953	0.244
$\mathbf{b}_{16}$	18.641	18.641	2239	3751	-0.004	0.035	0.036	0.955	0.248
$\mathbf{b}_{17}$	18.403	18.403	2065	3814	0.000	0.035	0.035	0.950	0.251
$\mathbf{b}_{18}$	18.665	18.665	2013	3929	0.002	0.033	0.034	0.951	0.250
$\mathbf{b}_{19}$	19.455	19.455	2039	4153	0.002	0.034	0.034	0.953	0.245
$\mathbf{b}_{20}$	20.632	20.632	2094	4449	0.001	0.033	0.033	0.948	0.237
$\mathbf{b}_{21}$	21.791	21.791	2110	4669	0.000	0.033	0.033	0.950	0.232
`Uniform`								0.946	0.373
`WBATE`					-0.000	0.017	0.017	0.948	0.093
`LBATE`					0.043	0.027	0.050	0.984	0.390

Table SA-1: Sharp BATEC Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.845	18.845	2020	3988	-0.000	0.034	0.034	0.958	0.247
$\mathbf{b}_2$	17.539	17.539	1876	3510	-0.000	0.035	0.035	0.953	0.261
$\mathbf{b}_3$	15.909	15.909	1661	2913	0.000	0.038	0.038	0.948	0.285
$\mathbf{b}_4$	13.996	13.996	1382	2256	0.000	0.043	0.043	0.950	0.322
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.000	0.050	0.050	0.947	0.375
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.000	0.060	0.060	0.947	0.451
$\mathbf{b}_7$	8.547	8.547	594	781	0.000	0.071	0.071	0.942	0.532
$\mathbf{b}_8$	8.365	8.365	597	678	0.001	0.075	0.075	0.939	0.559
$\mathbf{b}_9$	8.380	8.380	641	580	0.002	0.077	0.077	0.942	0.572
$\mathbf{b}_{10}$	8.489	8.489	701	479	0.002	0.080	0.080	0.942	0.586
$\mathbf{b}_{11}$	9.316	9.316	871	442	-0.000	0.086	0.086	0.931	0.636
$\mathbf{b}_{12}$	8.429	8.429	675	519	-0.005	0.078	0.078	0.939	0.580
$\mathbf{b}_{13}$	8.414	8.414	612	667	-0.005	0.075	0.075	0.940	0.557
$\mathbf{b}_{14}$	8.724	8.724	615	822	-0.002	0.070	0.070	0.942	0.522
$\mathbf{b}_{15}$	11.200	11.200	961	1419	-0.001	0.054	0.054	0.941	0.402
$\mathbf{b}_{16}$	13.996	13.996	1382	2258	-0.000	0.043	0.043	0.942	0.322
$\mathbf{b}_{17}$	16.487	16.487	1740	3124	0.000	0.037	0.037	0.946	0.275
$\mathbf{b}_{18}$	18.113	18.113	1931	3712	0.001	0.034	0.034	0.951	0.256
$\mathbf{b}_{19}$	19.403	19.403	2033	4132	0.002	0.034	0.034	0.953	0.245
$\mathbf{b}_{20}$	20.632	20.632	2094	4449	0.001	0.033	0.033	0.948	0.237
$\mathbf{b}_{21}$	21.791	21.791	2110	4669	0.000	0.033	0.033	0.950	0.232
`Uniform`								0.934	0.616
`WBATE`					-0.000	0.021	0.021	0.952	0.120
`LBATE`					0.089	0.046	0.100	0.973	0.757

Table SA-2: Sharp BATEC Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.557	8.557	609	823	0.000	0.069	0.069	0.957	0.276
$\mathbf{b}_2$	8.382	8.382	597	801	-0.000	0.072	0.072	0.950	0.280
$\mathbf{b}_3$	8.271	8.271	588	786	-0.001	0.074	0.074	0.949	0.282
$\mathbf{b}_4$	8.229	8.229	582	777	-0.001	0.073	0.073	0.956	0.284
$\mathbf{b}_5$	8.248	8.248	579	772	-0.002	0.074	0.074	0.952	0.283
$\mathbf{b}_6$	8.314	8.314	576	770	-0.000	0.073	0.073	0.952	0.283
$\mathbf{b}_7$	8.364	8.364	574	750	-0.000	0.073	0.073	0.951	0.285
$\mathbf{b}_8$	8.365	8.365	597	678	0.001	0.075	0.075	0.951	0.291
$\mathbf{b}_9$	8.380	8.380	641	580	0.002	0.077	0.077	0.950	0.298
$\mathbf{b}_{10}$	8.489	8.489	701	479	0.002	0.080	0.080	0.951	0.309
$\mathbf{b}_{11}$	9.316	9.316	871	442	-0.000	0.086	0.086	0.947	0.330
$\mathbf{b}_{12}$	8.429	8.429	675	519	-0.005	0.078	0.078	0.950	0.303
$\mathbf{b}_{13}$	8.414	8.414	612	667	-0.005	0.075	0.075	0.953	0.291
$\mathbf{b}_{14}$	8.437	8.437	582	773	-0.002	0.074	0.074	0.949	0.282
$\mathbf{b}_{15}$	8.335	8.335	585	784	-0.001	0.074	0.074	0.945	0.281
$\mathbf{b}_{16}$	8.167	8.167	574	765	0.001	0.075	0.075	0.944	0.286
$\mathbf{b}_{17}$	8.063	8.063	561	746	0.001	0.075	0.075	0.947	0.289
$\mathbf{b}_{18}$	8.177	8.177	565	755	0.000	0.075	0.075	0.947	0.288
$\mathbf{b}_{19}$	8.523	8.523	588	795	0.002	0.073	0.073	0.947	0.282
$\mathbf{b}_{20}$	9.039	9.039	621	853	0.002	0.071	0.071	0.947	0.272
$\mathbf{b}_{21}$	9.547	9.547	640	896	0.001	0.069	0.069	0.945	0.266
`Uniform`								0.944	0.438
`WBATE`					-0.000	0.024	0.024	0.953	0.095
`LBATE`					0.123	0.049	0.132	0.981	0.457

Table SA-3: Sharp BATEC Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.346	19.346	2103	4191	-0.002	0.035	0.035	0.960	0.244
$\mathbf{b}_2$	19.187	19.187	2155	4130	-0.002	0.036	0.036	0.957	0.245
$\mathbf{b}_3$	19.183	19.183	2247	4066	-0.001	0.037	0.037	0.958	0.244
$\mathbf{b}_4$	19.301	19.301	2372	3985	0.001	0.037	0.037	0.959	0.244
$\mathbf{b}_5$	19.499	19.499	2514	3876	0.004	0.037	0.038	0.953	0.242
$\mathbf{b}_6$	19.771	19.771	2662	3744	0.006	0.037	0.037	0.953	0.241
$\mathbf{b}_7$	20.043	20.043	2791	3576	0.008	0.036	0.037	0.960	0.241
$\mathbf{b}_8$	20.290	20.290	2888	3372	0.007	0.036	0.037	0.958	0.243
$\mathbf{b}_9$	20.611	20.611	2967	3165	0.006	0.037	0.037	0.955	0.246
$\mathbf{b}_{10}$	21.431	21.431	3097	3076	0.003	0.037	0.038	0.957	0.248
$\mathbf{b}_{11}$	23.256	23.256	3339	3247	0.000	0.038	0.038	0.956	0.260
$\mathbf{b}_{12}$	21.129	21.129	3057	3118	-0.006	0.037	0.038	0.957	0.246
$\mathbf{b}_{13}$	20.504	20.504	2936	3371	-0.012	0.037	0.039	0.956	0.242
$\mathbf{b}_{14}$	20.155	20.155	2800	3662	-0.014	0.037	0.039	0.953	0.240
$\mathbf{b}_{15}$	19.764	19.764	2605	3882	-0.010	0.038	0.039	0.956	0.241
$\mathbf{b}_{16}$	19.102	19.102	2335	3923	-0.004	0.038	0.038	0.957	0.247
$\mathbf{b}_{17}$	18.429	18.429	2083	3831	0.001	0.037	0.037	0.956	0.254
$\mathbf{b}_{18}$	18.250	18.250	1960	3778	0.003	0.036	0.036	0.954	0.258
$\mathbf{b}_{19}$	18.549	18.549	1927	3806	0.003	0.036	0.036	0.952	0.257
$\mathbf{b}_{20}$	19.074	19.074	1909	3854	0.002	0.037	0.038	0.949	0.256
$\mathbf{b}_{21}$	19.668	19.668	1873	3873	0.001	0.038	0.038	0.949	0.258
`Uniform`								0.949	0.371
`WBATE`					-0.000	0.017	0.017	0.945	0.092
`LBATE`					0.047	0.031	0.057	0.983	0.409

Table SA-4: Sharp BATEC Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.917	19.917	2174	4411	-0.002	0.044	0.044	0.956	0.312
$\mathbf{b}_2$	19.329	19.329	2166	4181	-0.002	0.044	0.045	0.958	0.315
$\mathbf{b}_3$	18.839	18.839	2175	3939	-0.001	0.045	0.045	0.957	0.318
$\mathbf{b}_4$	18.488	18.488	2211	3699	0.001	0.046	0.046	0.953	0.320
$\mathbf{b}_5$	18.264	18.264	2270	3459	0.003	0.046	0.046	0.954	0.321
$\mathbf{b}_6$	18.133	18.133	2338	3215	0.006	0.046	0.046	0.958	0.323
$\mathbf{b}_7$	18.010	18.010	2394	2947	0.008	0.046	0.046	0.960	0.325
$\mathbf{b}_8$	17.712	17.712	2394	2613	0.008	0.047	0.047	0.956	0.332
$\mathbf{b}_9$	17.533	17.533	2395	2303	0.007	0.048	0.048	0.949	0.339
$\mathbf{b}_{10}$	17.461	17.461	2393	2015	0.005	0.048	0.049	0.954	0.343
$\mathbf{b}_{11}$	18.552	18.552	2578	1984	0.002	0.048	0.048	0.949	0.340
$\mathbf{b}_{12}$	17.414	17.414	2382	2113	-0.004	0.050	0.050	0.953	0.351
$\mathbf{b}_{13}$	17.561	17.561	2372	2520	-0.010	0.050	0.051	0.959	0.351
$\mathbf{b}_{14}$	17.865	17.865	2349	2953	-0.011	0.051	0.052	0.953	0.349
$\mathbf{b}_{15}$	17.984	17.984	2251	3296	-0.008	0.052	0.053	0.957	0.354
$\mathbf{b}_{16}$	18.195	18.195	2153	3603	-0.003	0.053	0.053	0.956	0.361
$\mathbf{b}_{17}$	18.627	18.627	2099	3896	0.000	0.054	0.054	0.954	0.367
$\mathbf{b}_{18}$	19.308	19.308	2100	4180	0.002	0.054	0.054	0.952	0.372
$\mathbf{b}_{19}$	20.440	20.440	2160	4548	0.002	0.053	0.053	0.952	0.371
$\mathbf{b}_{20}$	21.899	21.899	2232	4967	0.001	0.053	0.053	0.951	0.370
$\mathbf{b}_{21}$	23.484	23.484	2276	5358	0.000	0.054	0.054	0.957	0.371
`Uniform`								0.950	0.516
`WBATE`					0.000	0.022	0.022	0.952	0.121
`LBATE`					0.066	0.038	0.076	0.982	0.553

Table SA-5: Sharp BATEC Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.062	19.062	2049	4073	-0.002	0.045	0.045	0.954	0.324
$\mathbf{b}_2$	17.606	17.606	1887	3536	-0.001	0.046	0.046	0.951	0.342
$\mathbf{b}_3$	15.908	15.908	1661	2914	-0.000	0.050	0.050	0.948	0.371
$\mathbf{b}_4$	13.995	13.995	1382	2256	-0.000	0.055	0.055	0.945	0.414
$\mathbf{b}_5$	12.000	12.000	1082	1641	0.000	0.063	0.063	0.943	0.477
$\mathbf{b}_6$	10.000	10.000	789	1112	-0.000	0.075	0.075	0.944	0.571
$\mathbf{b}_7$	8.241	8.241	553	727	-0.001	0.090	0.090	0.937	0.688
$\mathbf{b}_8$	7.762	7.762	511	593	0.002	0.099	0.099	0.942	0.736
$\mathbf{b}_9$	7.681	7.681	537	496	0.004	0.103	0.103	0.934	0.751
$\mathbf{b}_{10}$	7.650	7.650	574	392	0.006	0.102	0.102	0.938	0.769
$\mathbf{b}_{11}$	8.128	8.128	681	328	0.003	0.101	0.101	0.938	0.759
$\mathbf{b}_{12}$	7.629	7.629	554	431	-0.004	0.105	0.105	0.937	0.776
$\mathbf{b}_{13}$	7.694	7.694	508	569	-0.003	0.104	0.105	0.942	0.775
$\mathbf{b}_{14}$	8.450	8.450	578	772	0.001	0.095	0.095	0.941	0.710
$\mathbf{b}_{15}$	11.200	11.200	962	1418	-0.001	0.074	0.074	0.946	0.551
$\mathbf{b}_{16}$	13.992	13.992	1382	2256	-0.001	0.062	0.062	0.943	0.459
$\mathbf{b}_{17}$	16.564	16.564	1751	3153	-0.001	0.056	0.056	0.949	0.407
$\mathbf{b}_{18}$	18.582	18.582	1993	3894	0.001	0.054	0.054	0.950	0.384
$\mathbf{b}_{19}$	20.310	20.310	2144	4494	0.001	0.053	0.053	0.952	0.373
$\mathbf{b}_{20}$	21.890	21.890	2231	4963	0.001	0.053	0.053	0.951	0.370
$\mathbf{b}_{21}$	23.483	23.483	2276	5358	0.000	0.054	0.054	0.957	0.371
`Uniform`								0.927	0.829
`WBATE`					0.000	0.027	0.027	0.949	0.153
`LBATE`					0.127	0.061	0.141	0.967	1.012

Table SA-6: Sharp BATEC Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.726	8.726	630	856	-0.003	0.093	0.093	0.950	0.356
$\mathbf{b}_2$	8.468	8.468	608	818	-0.002	0.093	0.093	0.948	0.360
$\mathbf{b}_3$	8.253	8.253	585	781	0.000	0.094	0.094	0.949	0.365
$\mathbf{b}_4$	8.100	8.100	566	751	0.001	0.095	0.095	0.947	0.369
$\mathbf{b}_5$	8.002	8.002	549	725	0.000	0.096	0.096	0.949	0.371
$\mathbf{b}_6$	7.944	7.944	532	701	-0.001	0.096	0.096	0.954	0.374
$\mathbf{b}_7$	7.890	7.890	514	670	-0.001	0.096	0.096	0.957	0.377
$\mathbf{b}_8$	7.760	7.760	511	593	0.002	0.099	0.100	0.951	0.384
$\mathbf{b}_9$	7.681	7.681	537	496	0.004	0.103	0.103	0.941	0.393
$\mathbf{b}_{10}$	7.650	7.650	574	392	0.006	0.102	0.102	0.951	0.397
$\mathbf{b}_{11}$	8.128	8.128	681	328	0.003	0.101	0.101	0.954	0.395
$\mathbf{b}_{12}$	7.629	7.629	554	431	-0.004	0.105	0.105	0.949	0.406
$\mathbf{b}_{13}$	7.693	7.693	508	569	-0.003	0.105	0.105	0.955	0.405
$\mathbf{b}_{14}$	7.827	7.827	507	665	0.001	0.104	0.104	0.953	0.403
$\mathbf{b}_{15}$	7.879	7.879	530	698	0.000	0.105	0.105	0.952	0.406
$\mathbf{b}_{16}$	7.971	7.971	550	726	-0.003	0.106	0.106	0.947	0.411
$\mathbf{b}_{17}$	8.161	8.161	573	762	-0.003	0.107	0.107	0.951	0.416
$\mathbf{b}_{18}$	8.459	8.459	600	807	-0.000	0.109	0.109	0.945	0.419
$\mathbf{b}_{19}$	8.955	8.955	641	878	-0.001	0.109	0.109	0.947	0.416
$\mathbf{b}_{20}$	9.594	9.594	688	963	-0.002	0.107	0.107	0.946	0.412
$\mathbf{b}_{21}$	10.288	10.288	727	1044	-0.001	0.106	0.106	0.953	0.411
`Uniform`								0.942	0.600
`WBATE`					-0.000	0.032	0.032	0.952	0.124
`LBATE`					0.175	0.069	0.188	0.979	0.636

Table SA-7: Sharp BATEC Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.141	19.141	2074	4119	-0.003	0.049	0.049	0.958	0.328
$\mathbf{b}_2$	18.748	18.748	2087	3973	-0.003	0.049	0.049	0.955	0.329
$\mathbf{b}_3$	18.546	18.546	2138	3846	-0.002	0.050	0.050	0.959	0.328
$\mathbf{b}_4$	18.409	18.409	2206	3689	-0.001	0.051	0.051	0.954	0.327
$\mathbf{b}_5$	18.352	18.352	2291	3510	0.002	0.051	0.051	0.954	0.326
$\mathbf{b}_6$	18.342	18.342	2380	3306	0.004	0.050	0.051	0.964	0.325
$\mathbf{b}_7$	17.995	17.995	2387	2973	0.006	0.050	0.050	0.963	0.331
$\mathbf{b}_8$	17.794	17.794	2404	2669	0.007	0.051	0.052	0.957	0.337
$\mathbf{b}_9$	17.707	17.707	2421	2387	0.006	0.053	0.053	0.954	0.343
$\mathbf{b}_{10}$	17.845	17.845	2456	2148	0.004	0.053	0.053	0.957	0.344
$\mathbf{b}_{11}$	18.868	18.868	2619	2107	0.001	0.052	0.052	0.957	0.341
$\mathbf{b}_{12}$	17.862	17.862	2459	2264	-0.004	0.054	0.054	0.956	0.350
$\mathbf{b}_{13}$	18.205	18.205	2495	2731	-0.009	0.055	0.055	0.960	0.347
$\mathbf{b}_{14}$	18.657	18.657	2506	3220	-0.010	0.055	0.056	0.954	0.343
$\mathbf{b}_{15}$	18.922	18.922	2442	3620	-0.007	0.057	0.057	0.960	0.347
$\mathbf{b}_{16}$	18.870	18.870	2297	3851	-0.002	0.058	0.058	0.960	0.359
$\mathbf{b}_{17}$	18.966	18.966	2179	4030	0.001	0.060	0.060	0.959	0.370
$\mathbf{b}_{18}$	19.187	19.187	2096	4149	0.003	0.060	0.061	0.958	0.381
$\mathbf{b}_{19}$	19.676	19.676	2065	4268	0.004	0.061	0.061	0.957	0.390
$\mathbf{b}_{20}$	20.508	20.508	2065	4446	0.003	0.063	0.063	0.953	0.399
$\mathbf{b}_{21}$	21.347	21.347	2041	4551	0.004	0.065	0.065	0.959	0.411
`Uniform`								0.949	0.528
`WBATE`					0.000	0.023	0.023	0.947	0.122
`LBATE`					0.080	0.052	0.095	0.980	0.616

Table SA-8: Sharp BATEC Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.917	17.917	1899	3638	-0.002	0.036	0.036	0.948	0.260
$\mathbf{b}_2$	17.386	17.386	1869	3461	-0.002	0.037	0.037	0.950	0.266
$\mathbf{b}_3$	17.186	17.186	1889	3364	-0.001	0.038	0.038	0.957	0.268
$\mathbf{b}_4$	17.352	17.352	1993	3331	0.002	0.038	0.039	0.957	0.265
$\mathbf{b}_5$	17.854	17.854	2186	3338	0.009	0.038	0.039	0.953	0.259
$\mathbf{b}_6$	18.432	18.432	2398	3310	0.016	0.037	0.041	0.950	0.253
$\mathbf{b}_7$	18.885	18.885	2573	3201	0.021	0.036	0.042	0.958	0.250
$\mathbf{b}_8$	19.002	19.002	2655	2966	0.022	0.035	0.041	0.952	0.253
$\mathbf{b}_9$	19.131	19.131	2712	2716	0.017	0.036	0.040	0.950	0.258
$\mathbf{b}_{10}$	19.388	19.388	2760	2485	0.008	0.038	0.039	0.951	0.267
$\mathbf{b}_{11}$	21.252	21.252	3048	2650	-0.002	0.038	0.038	0.947	0.279
$\mathbf{b}_{12}$	19.198	19.198	2728	2557	-0.010	0.037	0.039	0.947	0.263
$\mathbf{b}_{13}$	19.022	19.022	2668	2915	-0.017	0.036	0.040	0.954	0.254
$\mathbf{b}_{14}$	18.697	18.697	2520	3197	-0.018	0.037	0.041	0.951	0.252
$\mathbf{b}_{15}$	17.741	17.741	2201	3228	-0.012	0.039	0.041	0.948	0.261
$\mathbf{b}_{16}$	16.747	16.747	1880	3135	-0.004	0.040	0.040	0.956	0.274
$\mathbf{b}_{17}$	16.235	16.235	1717	3042	-0.001	0.039	0.039	0.947	0.282
$\mathbf{b}_{18}$	16.400	16.400	1699	3076	-0.001	0.038	0.038	0.945	0.282
$\mathbf{b}_{19}$	17.082	17.082	1739	3239	-0.002	0.037	0.037	0.946	0.276
$\mathbf{b}_{20}$	17.979	17.979	1779	3429	-0.003	0.037	0.037	0.947	0.269
$\mathbf{b}_{21}$	18.924	18.924	1793	3580	-0.005	0.036	0.037	0.952	0.265
`Uniform`								0.948	0.398
`WBATE`					0.001	0.017	0.017	0.950	0.095
`LBATE`					0.031	0.029	0.043	0.990	0.417

Table SA-9: Sharp BATEC Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.644	17.644	1860	3529	-0.002	0.036	0.036	0.948	0.263
$\mathbf{b}_2$	16.703	16.703	1759	3205	-0.001	0.038	0.038	0.948	0.274
$\mathbf{b}_3$	15.611	15.611	1617	2813	-0.001	0.039	0.039	0.951	0.290
$\mathbf{b}_4$	13.979	13.979	1378	2252	-0.001	0.043	0.043	0.949	0.322
$\mathbf{b}_5$	12.000	12.000	1080	1642	-0.001	0.050	0.050	0.945	0.375
$\mathbf{b}_6$	10.000	10.000	787	1114	-0.001	0.059	0.059	0.947	0.450
$\mathbf{b}_7$	8.509	8.509	589	775	-0.001	0.070	0.070	0.944	0.535
$\mathbf{b}_8$	8.325	8.325	592	674	0.003	0.075	0.075	0.942	0.559
$\mathbf{b}_9$	8.381	8.381	641	581	0.009	0.076	0.077	0.940	0.571
$\mathbf{b}_{10}$	8.494	8.494	702	480	0.007	0.080	0.080	0.942	0.585
$\mathbf{b}_{11}$	9.311	9.311	870	442	-0.001	0.086	0.086	0.933	0.636
$\mathbf{b}_{12}$	8.411	8.411	672	517	-0.007	0.079	0.079	0.942	0.580
$\mathbf{b}_{13}$	8.334	8.334	601	656	-0.004	0.076	0.076	0.941	0.562
$\mathbf{b}_{14}$	8.644	8.644	605	807	-0.000	0.070	0.070	0.943	0.524
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.053	0.053	0.945	0.401
$\mathbf{b}_{16}$	13.955	13.955	1376	2245	-0.002	0.044	0.044	0.947	0.322
$\mathbf{b}_{17}$	15.651	15.651	1620	2831	-0.001	0.040	0.040	0.944	0.290
$\mathbf{b}_{18}$	16.293	16.293	1684	3034	-0.001	0.038	0.038	0.945	0.283
$\mathbf{b}_{19}$	17.074	17.074	1738	3236	-0.002	0.037	0.037	0.946	0.276
$\mathbf{b}_{20}$	17.979	17.979	1779	3429	-0.003	0.037	0.037	0.947	0.269
$\mathbf{b}_{21}$	18.924	18.924	1793	3580	-0.005	0.036	0.037	0.952	0.265
`Uniform`								0.930	0.629
`WBATE`					-0.001	0.021	0.021	0.947	0.121
`LBATE`					0.085	0.054	0.100	0.981	0.792

Table SA-10: Sharp BATEC Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.849	7.849	524	693	0.001	0.078	0.078	0.945	0.300
$\mathbf{b}_2$	7.617	7.617	504	662	0.002	0.079	0.079	0.949	0.307
$\mathbf{b}_3$	7.529	7.529	498	651	0.001	0.079	0.079	0.948	0.310
$\mathbf{b}_4$	7.602	7.602	506	663	-0.001	0.080	0.080	0.946	0.307
$\mathbf{b}_5$	7.822	7.822	527	696	-0.002	0.078	0.078	0.945	0.300
$\mathbf{b}_6$	8.075	8.075	547	729	-0.001	0.075	0.075	0.954	0.292
$\mathbf{b}_7$	8.274	8.274	563	736	-0.001	0.073	0.073	0.954	0.288
$\mathbf{b}_8$	8.325	8.325	592	674	0.003	0.075	0.075	0.950	0.292
$\mathbf{b}_9$	8.381	8.381	641	581	0.009	0.076	0.077	0.951	0.298
$\mathbf{b}_{10}$	8.494	8.494	702	480	0.007	0.080	0.080	0.946	0.308
$\mathbf{b}_{11}$	9.311	9.311	870	442	-0.001	0.086	0.086	0.947	0.329
$\mathbf{b}_{12}$	8.411	8.411	672	517	-0.007	0.079	0.079	0.946	0.303
$\mathbf{b}_{13}$	8.333	8.333	601	656	-0.004	0.076	0.076	0.951	0.293
$\mathbf{b}_{14}$	8.191	8.191	553	729	-0.000	0.075	0.075	0.947	0.290
$\mathbf{b}_{15}$	7.772	7.772	519	682	0.000	0.078	0.078	0.949	0.302
$\mathbf{b}_{16}$	7.337	7.337	476	616	0.001	0.082	0.082	0.945	0.317
$\mathbf{b}_{17}$	7.113	7.113	449	577	0.001	0.086	0.086	0.944	0.327
$\mathbf{b}_{18}$	7.185	7.185	449	579	0.001	0.084	0.084	0.945	0.327
$\mathbf{b}_{19}$	7.484	7.484	468	609	-0.001	0.084	0.084	0.943	0.319
$\mathbf{b}_{20}$	7.877	7.877	488	645	-0.002	0.080	0.080	0.947	0.311
$\mathbf{b}_{21}$	8.291	8.291	502	673	-0.001	0.078	0.078	0.948	0.306
`Uniform`								0.937	0.467
`WBATE`					0.000	0.025	0.025	0.948	0.098
`LBATE`					0.114	0.052	0.126	0.988	0.488

Table SA-11: Sharp BATEC Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.226	17.226	1805	3381	-0.004	0.038	0.038	0.951	0.271
$\mathbf{b}_2$	16.871	16.871	1794	3282	-0.004	0.039	0.039	0.955	0.275
$\mathbf{b}_3$	16.770	16.770	1829	3228	-0.003	0.040	0.040	0.963	0.277
$\mathbf{b}_4$	16.999	16.999	1940	3229	-0.000	0.041	0.041	0.961	0.275
$\mathbf{b}_5$	17.517	17.517	2124	3259	0.006	0.042	0.043	0.956	0.270
$\mathbf{b}_6$	18.251	18.251	2356	3293	0.013	0.042	0.044	0.955	0.262
$\mathbf{b}_7$	18.950	18.950	2571	3265	0.018	0.040	0.044	0.956	0.256
$\mathbf{b}_8$	19.610	19.610	2754	3184	0.019	0.038	0.043	0.953	0.252
$\mathbf{b}_9$	20.292	20.292	2907	3079	0.016	0.037	0.040	0.950	0.249
$\mathbf{b}_{10}$	21.262	21.262	3066	3032	0.008	0.038	0.039	0.954	0.250
$\mathbf{b}_{11}$	23.045	23.045	3306	3188	-0.002	0.039	0.039	0.953	0.262
$\mathbf{b}_{12}$	20.859	20.859	3010	3043	-0.011	0.038	0.039	0.956	0.248
$\mathbf{b}_{13}$	19.881	19.881	2820	3194	-0.017	0.038	0.041	0.954	0.248
$\mathbf{b}_{14}$	18.689	18.689	2509	3233	-0.017	0.041	0.044	0.951	0.258
$\mathbf{b}_{15}$	17.167	17.167	2091	3088	-0.009	0.045	0.045	0.956	0.276
$\mathbf{b}_{16}$	15.870	15.870	1740	2869	-0.001	0.044	0.044	0.957	0.294
$\mathbf{b}_{17}$	15.296	15.296	1578	2731	0.002	0.043	0.043	0.948	0.302
$\mathbf{b}_{18}$	15.414	15.414	1561	2731	0.001	0.041	0.041	0.947	0.300
$\mathbf{b}_{19}$	15.959	15.959	1591	2833	0.000	0.041	0.041	0.949	0.294
$\mathbf{b}_{20}$	16.585	16.585	1605	2925	-0.001	0.041	0.041	0.955	0.290
$\mathbf{b}_{21}$	17.091	17.091	1577	2934	-0.003	0.042	0.042	0.955	0.292
`Uniform`								0.947	0.409
`WBATE`					0.001	0.018	0.018	0.949	0.096
`LBATE`					0.037	0.034	0.050	0.990	0.447

Table SA-12: Sharp BATEC Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.550	18.550	1986	3881	-0.002	0.042	0.042	0.949	0.302
$\mathbf{b}_2$	17.907	17.907	1948	3654	-0.001	0.042	0.042	0.955	0.304
$\mathbf{b}_3$	17.562	17.562	1954	3494	0.000	0.042	0.042	0.960	0.303
$\mathbf{b}_4$	17.483	17.483	2019	3373	0.003	0.043	0.043	0.960	0.299
$\mathbf{b}_5$	17.596	17.596	2134	3256	0.009	0.042	0.043	0.958	0.293
$\mathbf{b}_6$	17.768	17.768	2262	3110	0.016	0.042	0.045	0.958	0.287
$\mathbf{b}_7$	17.881	17.881	2367	2912	0.021	0.041	0.046	0.951	0.283
$\mathbf{b}_8$	17.656	17.656	2382	2600	0.021	0.041	0.046	0.950	0.287
$\mathbf{b}_9$	17.529	17.529	2394	2304	0.017	0.041	0.045	0.949	0.291
$\mathbf{b}_{10}$	17.409	17.409	2382	2004	0.009	0.042	0.043	0.952	0.294
$\mathbf{b}_{11}$	18.492	18.492	2566	1971	-0.001	0.042	0.042	0.951	0.293
$\mathbf{b}_{12}$	17.353	17.353	2369	2100	-0.009	0.043	0.044	0.956	0.300
$\mathbf{b}_{13}$	17.438	17.438	2346	2490	-0.016	0.043	0.046	0.954	0.298
$\mathbf{b}_{14}$	17.577	17.577	2288	2874	-0.017	0.043	0.046	0.952	0.296
$\mathbf{b}_{15}$	17.244	17.244	2100	3076	-0.010	0.045	0.046	0.957	0.305
$\mathbf{b}_{16}$	16.769	16.769	1884	3142	-0.003	0.046	0.046	0.962	0.318
$\mathbf{b}_{17}$	16.620	16.620	1777	3181	-0.000	0.046	0.046	0.954	0.328
$\mathbf{b}_{18}$	17.013	17.013	1784	3306	-0.001	0.046	0.046	0.947	0.329
$\mathbf{b}_{19}$	17.833	17.833	1836	3526	-0.002	0.045	0.045	0.953	0.325
$\mathbf{b}_{20}$	18.869	18.869	1887	3768	-0.004	0.045	0.045	0.954	0.321
$\mathbf{b}_{21}$	20.049	20.049	1920	4002	-0.006	0.045	0.045	0.945	0.318
`Uniform`								0.952	0.458
`WBATE`					0.001	0.019	0.019	0.950	0.105
`LBATE`					0.042	0.033	0.053	0.988	0.481

Table SA-13: Sharp BATEC Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.127	18.127	1925	3713	-0.001	0.043	0.043	0.950	0.307
$\mathbf{b}_2$	16.987	16.987	1800	3309	-0.000	0.044	0.043	0.950	0.317
$\mathbf{b}_3$	15.693	15.693	1630	2842	0.001	0.045	0.045	0.955	0.333
$\mathbf{b}_4$	13.973	13.973	1379	2251	0.001	0.048	0.048	0.948	0.366
$\mathbf{b}_5$	12.000	12.000	1082	1642	0.000	0.055	0.055	0.945	0.418
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.001	0.065	0.065	0.942	0.494
$\mathbf{b}_7$	8.216	8.216	549	723	0.001	0.081	0.081	0.940	0.595
$\mathbf{b}_8$	7.739	7.739	508	590	0.005	0.088	0.088	0.934	0.634
$\mathbf{b}_9$	7.680	7.680	536	496	0.011	0.088	0.088	0.943	0.641
$\mathbf{b}_{10}$	7.627	7.627	570	390	0.008	0.087	0.087	0.946	0.651
$\mathbf{b}_{11}$	8.101	8.101	677	326	-0.001	0.090	0.090	0.937	0.648
$\mathbf{b}_{12}$	7.602	7.602	550	428	-0.006	0.091	0.091	0.937	0.659
$\mathbf{b}_{13}$	7.640	7.640	500	562	-0.003	0.089	0.089	0.944	0.651
$\mathbf{b}_{14}$	8.433	8.433	576	770	-0.002	0.080	0.080	0.943	0.590
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.060	0.060	0.946	0.452
$\mathbf{b}_{16}$	13.943	13.943	1374	2241	-0.001	0.049	0.049	0.951	0.373
$\mathbf{b}_{17}$	15.831	15.831	1646	2896	-0.001	0.047	0.047	0.951	0.340
$\mathbf{b}_{18}$	16.848	16.848	1760	3241	-0.001	0.046	0.046	0.948	0.331
$\mathbf{b}_{19}$	17.819	17.819	1834	3520	-0.002	0.045	0.045	0.953	0.326
$\mathbf{b}_{20}$	18.869	18.869	1887	3768	-0.004	0.045	0.045	0.954	0.321
$\mathbf{b}_{21}$	20.049	20.049	1920	4002	-0.006	0.045	0.045	0.945	0.318
`Uniform`								0.932	0.713
`WBATE`					-0.000	0.023	0.023	0.950	0.131
`LBATE`					0.102	0.058	0.117	0.979	0.888

Table SA-14: Sharp BATEC Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.127	8.127	558	743	0.001	0.090	0.090	0.948	0.347
$\mathbf{b}_2$	7.845	7.845	533	703	0.001	0.090	0.090	0.946	0.350
$\mathbf{b}_3$	7.694	7.694	518	680	0.001	0.088	0.088	0.953	0.349
$\mathbf{b}_4$	7.659	7.659	513	673	0.000	0.088	0.088	0.947	0.345
$\mathbf{b}_5$	7.709	7.709	514	674	-0.002	0.087	0.087	0.952	0.338
$\mathbf{b}_6$	7.785	7.785	513	674	-0.002	0.086	0.086	0.951	0.332
$\mathbf{b}_7$	7.834	7.834	507	661	0.000	0.086	0.086	0.947	0.328
$\mathbf{b}_8$	7.735	7.735	508	590	0.005	0.088	0.088	0.947	0.331
$\mathbf{b}_9$	7.680	7.680	536	496	0.011	0.088	0.088	0.948	0.335
$\mathbf{b}_{10}$	7.627	7.627	570	390	0.008	0.087	0.087	0.950	0.338
$\mathbf{b}_{11}$	8.101	8.101	677	326	-0.001	0.090	0.090	0.946	0.339
$\mathbf{b}_{12}$	7.602	7.602	550	428	-0.006	0.091	0.091	0.945	0.345
$\mathbf{b}_{13}$	7.640	7.640	500	562	-0.003	0.089	0.089	0.950	0.342
$\mathbf{b}_{14}$	7.701	7.701	493	645	-0.002	0.088	0.088	0.951	0.341
$\mathbf{b}_{15}$	7.555	7.555	492	643	-0.000	0.089	0.089	0.948	0.348
$\mathbf{b}_{16}$	7.347	7.347	476	618	-0.001	0.093	0.093	0.946	0.362
$\mathbf{b}_{17}$	7.281	7.281	468	606	0.001	0.096	0.096	0.945	0.372
$\mathbf{b}_{18}$	7.453	7.453	480	625	-0.001	0.097	0.097	0.941	0.372
$\mathbf{b}_{19}$	7.813	7.813	505	667	-0.002	0.094	0.094	0.950	0.367
$\mathbf{b}_{20}$	8.267	8.267	533	714	0.000	0.093	0.093	0.947	0.361
$\mathbf{b}_{21}$	8.784	8.784	556	759	0.000	0.094	0.094	0.941	0.357
`Uniform`								0.943	0.531
`WBATE`					0.000	0.028	0.028	0.947	0.107
`LBATE`					0.137	0.059	0.149	0.986	0.556

Table SA-15: Sharp BATEC Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.834	17.834	1890	3614	-0.003	0.045	0.045	0.951	0.315
$\mathbf{b}_2$	17.392	17.392	1874	3473	-0.003	0.045	0.045	0.953	0.315
$\mathbf{b}_3$	17.123	17.123	1890	3353	-0.002	0.046	0.046	0.959	0.314
$\mathbf{b}_4$	17.070	17.070	1956	3254	0.001	0.047	0.047	0.964	0.311
$\mathbf{b}_5$	17.206	17.206	2067	3162	0.006	0.048	0.048	0.959	0.306
$\mathbf{b}_6$	17.450	17.450	2199	3053	0.011	0.048	0.049	0.958	0.299
$\mathbf{b}_7$	17.514	17.514	2288	2849	0.016	0.047	0.050	0.949	0.296
$\mathbf{b}_8$	17.563	17.563	2356	2616	0.018	0.047	0.050	0.948	0.295
$\mathbf{b}_9$	17.567	17.567	2393	2354	0.015	0.046	0.048	0.954	0.296
$\mathbf{b}_{10}$	17.679	17.679	2426	2107	0.008	0.045	0.046	0.956	0.296
$\mathbf{b}_{11}$	18.549	18.549	2564	2029	-0.002	0.046	0.046	0.958	0.298
$\mathbf{b}_{12}$	17.600	17.600	2410	2197	-0.009	0.047	0.048	0.957	0.302
$\mathbf{b}_{13}$	17.641	17.641	2383	2579	-0.014	0.047	0.050	0.956	0.301
$\mathbf{b}_{14}$	17.575	17.575	2288	2913	-0.013	0.049	0.051	0.953	0.303
$\mathbf{b}_{15}$	16.976	16.976	2057	3031	-0.006	0.051	0.052	0.962	0.318
$\mathbf{b}_{16}$	16.166	16.166	1798	2969	0.001	0.050	0.050	0.961	0.337
$\mathbf{b}_{17}$	15.779	15.779	1656	2908	0.004	0.051	0.051	0.960	0.349
$\mathbf{b}_{18}$	15.962	15.962	1637	2940	0.003	0.051	0.051	0.950	0.351
$\mathbf{b}_{19}$	16.531	16.531	1665	3054	0.001	0.050	0.050	0.955	0.350
$\mathbf{b}_{20}$	17.189	17.189	1679	3159	0.000	0.051	0.051	0.954	0.351
$\mathbf{b}_{21}$	17.845	17.845	1664	3220	-0.000	0.053	0.053	0.947	0.356
`Uniform`								0.954	0.480
`WBATE`					0.001	0.020	0.020	0.947	0.107
`LBATE`					0.052	0.042	0.067	0.991	0.540

Table SA-16: Sharp BATEC Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.534	19.534	2120	4260	0.000	0.033	0.033	0.950	0.241
$\mathbf{b}_2$	19.152	19.152	2138	4114	0.000	0.034	0.034	0.952	0.244
$\mathbf{b}_3$	18.902	18.902	2185	3961	0.001	0.035	0.035	0.950	0.246
$\mathbf{b}_4$	18.805	18.805	2271	3805	0.002	0.035	0.036	0.952	0.247
$\mathbf{b}_5$	18.832	18.832	2383	3639	0.004	0.035	0.036	0.955	0.247
$\mathbf{b}_6$	19.003	19.003	2517	3478	0.006	0.035	0.036	0.958	0.247
$\mathbf{b}_7$	19.115	19.115	2621	3265	0.008	0.035	0.036	0.955	0.248
$\mathbf{b}_8$	19.116	19.116	2679	2996	0.008	0.036	0.037	0.953	0.253
$\mathbf{b}_9$	19.218	19.218	2729	2738	0.007	0.037	0.038	0.952	0.259
$\mathbf{b}_{10}$	19.480	19.480	2777	2508	0.004	0.038	0.038	0.952	0.268
$\mathbf{b}_{11}$	21.345	21.345	3064	2673	0.001	0.039	0.039	0.954	0.280
$\mathbf{b}_{12}$	19.319	19.319	2751	2587	-0.005	0.038	0.038	0.954	0.264
$\mathbf{b}_{13}$	19.199	19.199	2704	2962	-0.011	0.037	0.038	0.949	0.254
$\mathbf{b}_{14}$	19.204	19.204	2625	3342	-0.012	0.036	0.038	0.946	0.247
$\mathbf{b}_{15}$	18.993	18.993	2458	3608	-0.009	0.036	0.037	0.950	0.245
$\mathbf{b}_{16}$	18.625	18.625	2237	3746	-0.003	0.037	0.037	0.955	0.249
$\mathbf{b}_{17}$	18.421	18.421	2068	3821	0.001	0.035	0.035	0.959	0.252
$\mathbf{b}_{18}$	18.678	18.678	2014	3935	0.001	0.034	0.034	0.957	0.251
$\mathbf{b}_{19}$	19.490	19.490	2043	4167	0.001	0.033	0.033	0.949	0.244
$\mathbf{b}_{20}$	20.668	20.668	2098	4464	0.001	0.033	0.033	0.956	0.236
$\mathbf{b}_{21}$	21.765	21.765	2107	4658	-0.000	0.033	0.033	0.953	0.232
`Uniform`								0.950	0.374
`WBATE`					0.000	0.017	0.017	0.949	0.093
`LBATE`					0.044	0.027	0.052	0.982	0.391

Table SA-17: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.862	18.862	2022	3994	0.001	0.034	0.034	0.950	0.248
$\mathbf{b}_2$	17.546	17.546	1877	3512	0.001	0.036	0.036	0.947	0.262
$\mathbf{b}_3$	15.909	15.909	1661	2914	0.001	0.039	0.039	0.948	0.286
$\mathbf{b}_4$	13.997	13.997	1382	2257	0.001	0.044	0.044	0.949	0.324
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.000	0.050	0.050	0.943	0.378
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.002	0.059	0.059	0.943	0.454
$\mathbf{b}_7$	8.562	8.562	596	783	-0.002	0.071	0.071	0.947	0.535
$\mathbf{b}_8$	8.375	8.375	599	680	0.001	0.076	0.076	0.938	0.560
$\mathbf{b}_9$	8.420	8.420	647	585	0.004	0.077	0.077	0.944	0.574
$\mathbf{b}_{10}$	8.534	8.534	708	483	0.003	0.079	0.079	0.940	0.586
$\mathbf{b}_{11}$	9.351	9.351	877	445	-0.000	0.086	0.086	0.936	0.639
$\mathbf{b}_{12}$	8.464	8.464	680	522	-0.006	0.077	0.078	0.943	0.582
$\mathbf{b}_{13}$	8.411	8.411	612	667	-0.004	0.076	0.076	0.948	0.561
$\mathbf{b}_{14}$	8.717	8.717	614	821	0.001	0.071	0.071	0.939	0.524
$\mathbf{b}_{15}$	11.200	11.200	961	1419	0.002	0.054	0.054	0.943	0.403
$\mathbf{b}_{16}$	13.996	13.996	1382	2258	0.001	0.043	0.043	0.946	0.323
$\mathbf{b}_{17}$	16.483	16.483	1739	3123	0.000	0.037	0.037	0.949	0.277
$\mathbf{b}_{18}$	18.112	18.112	1930	3712	0.001	0.034	0.034	0.954	0.257
$\mathbf{b}_{19}$	19.440	19.440	2038	4146	0.001	0.033	0.033	0.949	0.245
$\mathbf{b}_{20}$	20.668	20.668	2098	4464	0.001	0.033	0.033	0.956	0.236
$\mathbf{b}_{21}$	21.765	21.765	2107	4658	-0.000	0.033	0.033	0.953	0.232
`Uniform`								0.929	0.618
`WBATE`					0.000	0.021	0.021	0.951	0.120
`LBATE`					0.091	0.048	0.103	0.967	0.752

Table SA-18: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.558	8.558	609	823	0.001	0.072	0.072	0.944	0.277
$\mathbf{b}_2$	8.391	8.391	598	803	0.001	0.073	0.073	0.942	0.281
$\mathbf{b}_3$	8.281	8.281	589	788	0.001	0.073	0.073	0.950	0.283
$\mathbf{b}_4$	8.239	8.239	583	779	0.001	0.074	0.074	0.948	0.285
$\mathbf{b}_5$	8.251	8.251	579	773	-0.001	0.073	0.073	0.948	0.285
$\mathbf{b}_6$	8.325	8.325	577	772	-0.002	0.073	0.073	0.951	0.285
$\mathbf{b}_7$	8.375	8.375	575	752	-0.002	0.073	0.073	0.952	0.286
$\mathbf{b}_8$	8.375	8.375	599	680	0.001	0.076	0.076	0.947	0.292
$\mathbf{b}_9$	8.420	8.420	647	585	0.004	0.077	0.077	0.951	0.299
$\mathbf{b}_{10}$	8.534	8.534	708	483	0.003	0.079	0.079	0.950	0.309
$\mathbf{b}_{11}$	9.351	9.351	877	445	-0.000	0.086	0.086	0.947	0.331
$\mathbf{b}_{12}$	8.464	8.464	680	522	-0.006	0.077	0.078	0.951	0.304
$\mathbf{b}_{13}$	8.411	8.411	612	667	-0.004	0.076	0.076	0.952	0.293
$\mathbf{b}_{14}$	8.413	8.413	579	768	0.001	0.075	0.075	0.945	0.284
$\mathbf{b}_{15}$	8.321	8.321	583	781	0.002	0.074	0.074	0.947	0.283
$\mathbf{b}_{16}$	8.160	8.160	573	764	0.001	0.074	0.074	0.950	0.287
$\mathbf{b}_{17}$	8.070	8.070	561	747	-0.000	0.075	0.075	0.947	0.291
$\mathbf{b}_{18}$	8.183	8.183	566	756	-0.001	0.074	0.074	0.952	0.289
$\mathbf{b}_{19}$	8.539	8.539	590	798	-0.000	0.074	0.074	0.942	0.281
$\mathbf{b}_{20}$	9.055	9.055	623	856	-0.000	0.070	0.070	0.952	0.271
$\mathbf{b}_{21}$	9.535	9.535	639	893	-0.001	0.069	0.069	0.950	0.266
`Uniform`								0.943	0.439
`WBATE`					-0.000	0.024	0.024	0.951	0.095
`LBATE`					0.124	0.049	0.134	0.980	0.457

Table SA-19: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.325	19.325	2100	4182	-0.001	0.036	0.036	0.953	0.246
$\mathbf{b}_2$	19.171	19.171	2152	4123	-0.000	0.036	0.036	0.952	0.246
$\mathbf{b}_3$	19.127	19.127	2235	4046	0.000	0.037	0.037	0.953	0.246
$\mathbf{b}_4$	19.198	19.198	2351	3949	0.001	0.037	0.037	0.955	0.246
$\mathbf{b}_5$	19.404	19.404	2495	3845	0.003	0.037	0.038	0.958	0.245
$\mathbf{b}_6$	19.638	19.638	2635	3703	0.005	0.037	0.038	0.962	0.244
$\mathbf{b}_7$	19.901	19.901	2763	3534	0.007	0.037	0.038	0.957	0.244
$\mathbf{b}_8$	20.219	20.219	2874	3351	0.007	0.038	0.039	0.954	0.245
$\mathbf{b}_9$	20.701	20.701	2983	3189	0.006	0.038	0.038	0.953	0.246
$\mathbf{b}_{10}$	21.556	21.556	3119	3110	0.004	0.038	0.038	0.953	0.249
$\mathbf{b}_{11}$	23.400	23.400	3362	3287	0.001	0.039	0.039	0.956	0.261
$\mathbf{b}_{12}$	21.216	21.216	3072	3143	-0.006	0.038	0.038	0.957	0.246
$\mathbf{b}_{13}$	20.499	20.499	2936	3369	-0.011	0.037	0.039	0.953	0.243
$\mathbf{b}_{14}$	20.118	20.118	2793	3651	-0.012	0.038	0.039	0.950	0.241
$\mathbf{b}_{15}$	19.695	19.695	2591	3862	-0.008	0.039	0.040	0.957	0.243
$\mathbf{b}_{16}$	19.024	19.024	2321	3897	-0.002	0.039	0.039	0.958	0.249
$\mathbf{b}_{17}$	18.414	18.414	2081	3827	0.002	0.038	0.038	0.962	0.256
$\mathbf{b}_{18}$	18.243	18.243	1959	3776	0.003	0.036	0.036	0.961	0.259
$\mathbf{b}_{19}$	18.530	18.530	1924	3797	0.002	0.036	0.036	0.950	0.257
$\mathbf{b}_{20}$	19.104	19.104	1912	3865	0.001	0.037	0.037	0.953	0.256
$\mathbf{b}_{21}$	19.617	19.617	1867	3854	0.000	0.038	0.038	0.953	0.258
`Uniform`								0.948	0.373
`WBATE`					0.000	0.017	0.017	0.946	0.093
`LBATE`					0.049	0.032	0.059	0.982	0.410

Table SA-20: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.822	19.822	2159	4374	-0.001	0.042	0.042	0.955	0.300
$\mathbf{b}_2$	19.210	19.210	2146	4137	-0.000	0.043	0.043	0.958	0.303
$\mathbf{b}_3$	18.737	18.737	2155	3903	0.000	0.043	0.043	0.959	0.306
$\mathbf{b}_4$	18.364	18.364	2185	3658	0.002	0.043	0.043	0.957	0.307
$\mathbf{b}_5$	18.091	18.091	2234	3405	0.003	0.044	0.044	0.954	0.309
$\mathbf{b}_6$	17.917	17.917	2293	3150	0.006	0.044	0.045	0.954	0.311
$\mathbf{b}_7$	17.759	17.759	2342	2876	0.007	0.045	0.046	0.952	0.313
$\mathbf{b}_8$	17.432	17.432	2336	2540	0.007	0.046	0.047	0.957	0.321
$\mathbf{b}_9$	17.290	17.290	2346	2244	0.006	0.047	0.047	0.950	0.327
$\mathbf{b}_{10}$	17.153	17.153	2333	1945	0.004	0.047	0.047	0.951	0.330
$\mathbf{b}_{11}$	18.116	18.116	2499	1885	0.001	0.046	0.046	0.950	0.322
$\mathbf{b}_{12}$	17.198	17.198	2339	2062	-0.005	0.048	0.048	0.953	0.340
$\mathbf{b}_{13}$	17.364	17.364	2332	2471	-0.010	0.049	0.050	0.954	0.342
$\mathbf{b}_{14}$	17.720	17.720	2319	2912	-0.011	0.050	0.052	0.952	0.340
$\mathbf{b}_{15}$	17.870	17.870	2228	3261	-0.008	0.052	0.053	0.947	0.347
$\mathbf{b}_{16}$	18.113	18.113	2137	3576	-0.003	0.052	0.052	0.952	0.354
$\mathbf{b}_{17}$	18.540	18.540	2085	3865	0.001	0.052	0.052	0.955	0.362
$\mathbf{b}_{18}$	19.281	19.281	2097	4170	0.002	0.053	0.053	0.951	0.367
$\mathbf{b}_{19}$	20.431	20.431	2159	4545	0.002	0.053	0.053	0.954	0.367
$\mathbf{b}_{20}$	21.957	21.957	2239	4992	0.001	0.053	0.053	0.955	0.366
$\mathbf{b}_{21}$	23.561	23.561	2283	5391	0.000	0.054	0.054	0.957	0.368
`Uniform`								0.946	0.502
`WBATE`					0.000	0.021	0.021	0.950	0.116
`LBATE`					0.065	0.038	0.075	0.979	0.539

Table SA-21: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.027	19.027	2044	4060	-0.000	0.043	0.043	0.952	0.311
$\mathbf{b}_2$	17.579	17.579	1882	3526	0.000	0.045	0.045	0.954	0.328
$\mathbf{b}_3$	15.902	15.902	1660	2912	0.001	0.047	0.047	0.953	0.354
$\mathbf{b}_4$	13.995	13.995	1382	2257	0.001	0.052	0.052	0.948	0.395
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.001	0.061	0.061	0.946	0.456
$\mathbf{b}_6$	10.000	10.000	788	1112	-0.002	0.072	0.072	0.945	0.542
$\mathbf{b}_7$	8.181	8.181	545	717	-0.003	0.089	0.089	0.945	0.660
$\mathbf{b}_8$	7.642	7.642	495	577	0.000	0.096	0.096	0.945	0.714
$\mathbf{b}_9$	7.575	7.575	521	483	0.003	0.101	0.101	0.934	0.725
$\mathbf{b}_{10}$	7.515	7.515	554	378	0.002	0.100	0.100	0.940	0.742
$\mathbf{b}_{11}$	7.937	7.937	653	311	-0.001	0.098	0.098	0.932	0.718
$\mathbf{b}_{12}$	7.534	7.534	540	421	-0.006	0.101	0.101	0.945	0.755
$\mathbf{b}_{13}$	7.608	7.608	496	557	-0.002	0.102	0.102	0.942	0.755
$\mathbf{b}_{14}$	8.434	8.434	576	769	0.000	0.093	0.093	0.938	0.688
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.074	0.074	0.939	0.535
$\mathbf{b}_{16}$	13.991	13.991	1382	2256	-0.000	0.061	0.061	0.941	0.448
$\mathbf{b}_{17}$	16.545	16.545	1748	3147	0.000	0.055	0.055	0.949	0.399
$\mathbf{b}_{18}$	18.559	18.559	1990	3886	0.001	0.054	0.054	0.949	0.378
$\mathbf{b}_{19}$	20.288	20.288	2142	4486	0.001	0.053	0.053	0.953	0.369
$\mathbf{b}_{20}$	21.947	21.947	2238	4988	0.001	0.053	0.053	0.955	0.366
$\mathbf{b}_{21}$	23.560	23.560	2283	5391	0.000	0.054	0.054	0.957	0.368
`Uniform`								0.924	0.802
`WBATE`					-0.000	0.026	0.026	0.946	0.147
`LBATE`					0.123	0.060	0.137	0.966	0.974

Table SA-22: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.684	8.684	625	848	-0.001	0.088	0.088	0.951	0.341
$\mathbf{b}_2$	8.416	8.416	601	807	0.000	0.088	0.088	0.951	0.346
$\mathbf{b}_3$	8.209	8.209	580	773	0.001	0.090	0.090	0.952	0.350
$\mathbf{b}_4$	8.045	8.045	559	740	-0.000	0.092	0.092	0.950	0.353
$\mathbf{b}_5$	7.926	7.926	539	711	-0.002	0.093	0.093	0.952	0.357
$\mathbf{b}_6$	7.850	7.850	521	684	-0.003	0.093	0.093	0.952	0.360
$\mathbf{b}_7$	7.780	7.780	501	651	-0.002	0.094	0.094	0.951	0.365
$\mathbf{b}_8$	7.637	7.637	495	576	0.000	0.096	0.096	0.952	0.373
$\mathbf{b}_9$	7.575	7.575	521	483	0.003	0.101	0.101	0.948	0.379
$\mathbf{b}_{10}$	7.515	7.515	554	378	0.002	0.100	0.100	0.944	0.383
$\mathbf{b}_{11}$	7.937	7.937	653	311	-0.001	0.098	0.098	0.948	0.373
$\mathbf{b}_{12}$	7.534	7.534	540	421	-0.006	0.101	0.101	0.952	0.394
$\mathbf{b}_{13}$	7.607	7.607	496	557	-0.002	0.102	0.102	0.950	0.395
$\mathbf{b}_{14}$	7.763	7.763	500	654	-0.000	0.103	0.103	0.947	0.394
$\mathbf{b}_{15}$	7.829	7.829	524	689	-0.002	0.105	0.105	0.945	0.397
$\mathbf{b}_{16}$	7.935	7.935	546	720	0.000	0.105	0.105	0.944	0.403
$\mathbf{b}_{17}$	8.122	8.122	568	755	0.001	0.105	0.105	0.952	0.409
$\mathbf{b}_{18}$	8.447	8.447	598	805	-0.000	0.108	0.108	0.946	0.412
$\mathbf{b}_{19}$	8.951	8.951	640	877	-0.000	0.107	0.107	0.947	0.411
$\mathbf{b}_{20}$	9.620	9.620	691	969	0.002	0.105	0.105	0.948	0.408
$\mathbf{b}_{21}$	10.322	10.322	731	1051	0.001	0.106	0.106	0.951	0.408
`Uniform`								0.938	0.583
`WBATE`					-0.001	0.031	0.031	0.946	0.120
`LBATE`					0.170	0.068	0.184	0.975	0.618

Table SA-23: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.920	18.920	2042	4036	-0.002	0.048	0.048	0.954	0.318
$\mathbf{b}_2$	18.614	18.614	2065	3924	-0.001	0.047	0.047	0.963	0.317
$\mathbf{b}_3$	18.397	18.397	2110	3794	-0.001	0.048	0.048	0.960	0.316
$\mathbf{b}_4$	18.284	18.284	2182	3647	-0.000	0.048	0.048	0.958	0.314
$\mathbf{b}_5$	18.172	18.172	2256	3454	0.002	0.049	0.049	0.955	0.314
$\mathbf{b}_6$	18.104	18.104	2332	3236	0.004	0.049	0.049	0.957	0.314
$\mathbf{b}_7$	17.715	17.715	2330	2898	0.005	0.051	0.051	0.954	0.321
$\mathbf{b}_8$	17.474	17.474	2340	2587	0.006	0.051	0.052	0.958	0.327
$\mathbf{b}_9$	17.306	17.306	2344	2288	0.005	0.053	0.053	0.955	0.334
$\mathbf{b}_{10}$	17.385	17.385	2370	2040	0.003	0.052	0.053	0.954	0.334
$\mathbf{b}_{11}$	18.282	18.282	2516	1968	0.000	0.051	0.051	0.954	0.326
$\mathbf{b}_{12}$	17.522	17.522	2395	2180	-0.005	0.052	0.053	0.955	0.341
$\mathbf{b}_{13}$	17.903	17.903	2435	2652	-0.010	0.054	0.055	0.958	0.340
$\mathbf{b}_{14}$	18.374	18.374	2449	3140	-0.010	0.055	0.056	0.959	0.337
$\mathbf{b}_{15}$	18.805	18.805	2421	3586	-0.006	0.057	0.058	0.955	0.340
$\mathbf{b}_{16}$	18.919	18.919	2307	3867	-0.002	0.058	0.058	0.956	0.350
$\mathbf{b}_{17}$	18.942	18.942	2175	4021	0.002	0.057	0.057	0.960	0.363
$\mathbf{b}_{18}$	19.136	19.136	2090	4131	0.003	0.060	0.060	0.958	0.376
$\mathbf{b}_{19}$	19.695	19.695	2068	4278	0.003	0.061	0.061	0.955	0.386
$\mathbf{b}_{20}$	20.557	20.557	2072	4464	0.004	0.061	0.062	0.955	0.395
$\mathbf{b}_{21}$	21.470	21.470	2054	4601	0.005	0.064	0.064	0.959	0.407
`Uniform`								0.949	0.515
`WBATE`					0.000	0.022	0.022	0.942	0.118
`LBATE`					0.079	0.052	0.094	0.979	0.604

Table SA-24: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.945	17.945	1903	3648	-0.002	0.035	0.035	0.948	0.262
$\mathbf{b}_2$	17.380	17.380	1868	3459	-0.002	0.036	0.036	0.950	0.268
$\mathbf{b}_3$	17.202	17.202	1891	3370	-0.001	0.037	0.037	0.959	0.269
$\mathbf{b}_4$	17.380	17.380	1997	3341	0.002	0.039	0.039	0.956	0.267
$\mathbf{b}_5$	17.857	17.857	2186	3339	0.009	0.039	0.040	0.954	0.260
$\mathbf{b}_6$	18.430	18.430	2397	3310	0.016	0.038	0.042	0.951	0.254
$\mathbf{b}_7$	18.889	18.889	2574	3202	0.021	0.037	0.043	0.948	0.251
$\mathbf{b}_8$	18.988	18.988	2653	2961	0.022	0.036	0.042	0.948	0.254
$\mathbf{b}_9$	19.108	19.108	2707	2710	0.017	0.036	0.040	0.947	0.259
$\mathbf{b}_{10}$	19.393	19.393	2761	2487	0.009	0.037	0.038	0.951	0.268
$\mathbf{b}_{11}$	21.303	21.303	3057	2662	-0.000	0.038	0.038	0.953	0.279
$\mathbf{b}_{12}$	19.219	19.219	2732	2561	-0.008	0.037	0.038	0.953	0.263
$\mathbf{b}_{13}$	19.004	19.004	2665	2911	-0.015	0.037	0.040	0.948	0.254
$\mathbf{b}_{14}$	18.692	18.692	2519	3196	-0.017	0.037	0.041	0.947	0.252
$\mathbf{b}_{15}$	17.796	17.796	2212	3246	-0.012	0.039	0.041	0.952	0.260
$\mathbf{b}_{16}$	16.806	16.806	1891	3154	-0.004	0.039	0.039	0.953	0.273
$\mathbf{b}_{17}$	16.262	16.262	1722	3052	-0.001	0.038	0.038	0.954	0.282
$\mathbf{b}_{18}$	16.380	16.380	1697	3070	-0.001	0.038	0.038	0.952	0.282
$\mathbf{b}_{19}$	17.047	17.047	1735	3229	-0.001	0.038	0.038	0.942	0.276
$\mathbf{b}_{20}$	17.934	17.934	1774	3415	-0.002	0.038	0.038	0.944	0.269
$\mathbf{b}_{21}$	18.901	18.901	1791	3574	-0.005	0.037	0.038	0.949	0.264
`Uniform`								0.947	0.398
`WBATE`					0.001	0.017	0.018	0.950	0.096
`LBATE`					0.032	0.029	0.043	0.988	0.416

Table SA-25: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.655	17.655	1862	3533	-0.002	0.036	0.036	0.949	0.265
$\mathbf{b}_2$	16.710	16.710	1760	3207	-0.001	0.037	0.037	0.950	0.275
$\mathbf{b}_3$	15.625	15.625	1619	2818	-0.001	0.039	0.039	0.954	0.291
$\mathbf{b}_4$	13.979	13.979	1379	2252	-0.000	0.043	0.043	0.948	0.324
$\mathbf{b}_5$	12.000	12.000	1080	1642	0.000	0.050	0.050	0.950	0.377
$\mathbf{b}_6$	10.000	10.000	787	1114	0.000	0.060	0.060	0.950	0.453
$\mathbf{b}_7$	8.509	8.509	589	775	-0.000	0.073	0.073	0.940	0.538
$\mathbf{b}_8$	8.319	8.319	591	673	0.004	0.077	0.077	0.944	0.563
$\mathbf{b}_9$	8.372	8.372	639	580	0.010	0.077	0.077	0.940	0.572
$\mathbf{b}_{10}$	8.496	8.496	702	480	0.009	0.078	0.079	0.942	0.586
$\mathbf{b}_{11}$	9.333	9.333	873	444	0.002	0.085	0.085	0.932	0.637
$\mathbf{b}_{12}$	8.420	8.420	673	518	-0.004	0.078	0.078	0.945	0.580
$\mathbf{b}_{13}$	8.326	8.326	600	655	-0.001	0.076	0.076	0.944	0.562
$\mathbf{b}_{14}$	8.643	8.643	604	807	0.001	0.070	0.070	0.947	0.525
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.054	0.054	0.945	0.401
$\mathbf{b}_{16}$	13.958	13.958	1377	2246	-0.002	0.043	0.043	0.946	0.322
$\mathbf{b}_{17}$	15.667	15.667	1623	2837	-0.002	0.039	0.039	0.952	0.290
$\mathbf{b}_{18}$	16.274	16.274	1681	3028	-0.001	0.038	0.038	0.953	0.283
$\mathbf{b}_{19}$	17.039	17.039	1734	3226	-0.001	0.038	0.038	0.942	0.276
$\mathbf{b}_{20}$	17.934	17.934	1774	3415	-0.002	0.038	0.038	0.944	0.269
$\mathbf{b}_{21}$	18.901	18.901	1791	3574	-0.005	0.037	0.038	0.949	0.264
`Uniform`								0.930	0.631
`WBATE`					0.000	0.021	0.021	0.949	0.121
`LBATE`					0.086	0.052	0.101	0.979	0.791

Table SA-26: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.862	7.862	525	695	0.001	0.079	0.079	0.944	0.302
$\mathbf{b}_2$	7.614	7.614	504	661	0.001	0.082	0.082	0.941	0.310
$\mathbf{b}_3$	7.536	7.536	498	652	-0.000	0.080	0.080	0.949	0.312
$\mathbf{b}_4$	7.614	7.614	507	666	-0.001	0.080	0.080	0.947	0.309
$\mathbf{b}_5$	7.823	7.823	527	697	-0.000	0.077	0.077	0.953	0.301
$\mathbf{b}_6$	8.074	8.074	547	729	-0.000	0.076	0.076	0.951	0.294
$\mathbf{b}_7$	8.275	8.275	563	736	-0.000	0.076	0.076	0.950	0.290
$\mathbf{b}_8$	8.319	8.319	591	673	0.004	0.077	0.077	0.945	0.293
$\mathbf{b}_9$	8.372	8.372	639	580	0.010	0.077	0.077	0.948	0.299
$\mathbf{b}_{10}$	8.496	8.496	702	480	0.009	0.078	0.079	0.954	0.309
$\mathbf{b}_{11}$	9.333	9.333	873	444	0.002	0.085	0.085	0.950	0.330
$\mathbf{b}_{12}$	8.420	8.420	673	518	-0.004	0.078	0.078	0.948	0.304
$\mathbf{b}_{13}$	8.326	8.326	600	655	-0.001	0.076	0.076	0.951	0.293
$\mathbf{b}_{14}$	8.189	8.189	553	729	0.001	0.075	0.075	0.947	0.290
$\mathbf{b}_{15}$	7.797	7.797	522	687	-0.001	0.078	0.078	0.946	0.301
$\mathbf{b}_{16}$	7.363	7.363	479	620	-0.001	0.083	0.083	0.943	0.316
$\mathbf{b}_{17}$	7.124	7.124	450	579	-0.004	0.084	0.084	0.944	0.327
$\mathbf{b}_{18}$	7.176	7.176	448	577	-0.001	0.082	0.082	0.949	0.326
$\mathbf{b}_{19}$	7.468	7.468	466	607	0.003	0.084	0.084	0.940	0.319
$\mathbf{b}_{20}$	7.857	7.857	486	643	0.000	0.082	0.082	0.940	0.311
$\mathbf{b}_{21}$	8.281	8.281	501	672	-0.001	0.078	0.078	0.946	0.305
`Uniform`								0.938	0.468
`WBATE`					0.001	0.025	0.025	0.946	0.098
`LBATE`					0.115	0.052	0.126	0.989	0.486

Table SA-27: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.303	17.303	1816	3409	-0.003	0.038	0.038	0.950	0.271
$\mathbf{b}_2$	16.910	16.910	1799	3295	-0.004	0.039	0.039	0.951	0.276
$\mathbf{b}_3$	16.825	16.825	1837	3247	-0.003	0.040	0.040	0.959	0.278
$\mathbf{b}_4$	17.058	17.058	1950	3248	0.000	0.042	0.042	0.963	0.276
$\mathbf{b}_5$	17.578	17.578	2135	3278	0.006	0.043	0.044	0.958	0.270
$\mathbf{b}_6$	18.313	18.313	2368	3312	0.013	0.043	0.045	0.954	0.262
$\mathbf{b}_7$	19.002	19.002	2581	3280	0.019	0.041	0.045	0.950	0.256
$\mathbf{b}_8$	19.652	19.652	2762	3196	0.020	0.039	0.044	0.944	0.252
$\mathbf{b}_9$	20.340	20.340	2915	3094	0.016	0.037	0.041	0.946	0.250
$\mathbf{b}_{10}$	21.331	21.331	3079	3051	0.008	0.037	0.038	0.954	0.250
$\mathbf{b}_{11}$	23.114	23.114	3319	3205	-0.001	0.038	0.038	0.957	0.262
$\mathbf{b}_{12}$	20.904	20.904	3020	3053	-0.009	0.037	0.038	0.958	0.248
$\mathbf{b}_{13}$	19.854	19.854	2815	3187	-0.015	0.039	0.042	0.954	0.249
$\mathbf{b}_{14}$	18.695	18.695	2510	3236	-0.015	0.041	0.044	0.951	0.258
$\mathbf{b}_{15}$	17.220	17.220	2101	3104	-0.008	0.044	0.045	0.958	0.275
$\mathbf{b}_{16}$	15.920	15.920	1748	2886	-0.000	0.044	0.044	0.956	0.293
$\mathbf{b}_{17}$	15.326	15.326	1584	2741	0.001	0.041	0.041	0.952	0.301
$\mathbf{b}_{18}$	15.400	15.400	1560	2726	0.001	0.041	0.041	0.954	0.299
$\mathbf{b}_{19}$	15.907	15.907	1584	2816	0.001	0.042	0.042	0.946	0.294
$\mathbf{b}_{20}$	16.537	16.537	1600	2910	-0.001	0.043	0.043	0.949	0.290
$\mathbf{b}_{21}$	17.049	17.049	1573	2921	-0.003	0.043	0.043	0.955	0.291
`Uniform`								0.948	0.409
`WBATE`					0.001	0.018	0.018	0.946	0.096
`LBATE`					0.037	0.033	0.050	0.989	0.444

Table SA-28: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.213	18.213	1939	3752	-0.004	0.040	0.040	0.953	0.286
$\mathbf{b}_2$	17.557	17.557	1894	3525	-0.003	0.040	0.040	0.956	0.289
$\mathbf{b}_3$	17.203	17.203	1893	3370	-0.002	0.041	0.041	0.958	0.289
$\mathbf{b}_4$	17.159	17.159	1958	3268	0.002	0.042	0.042	0.953	0.285
$\mathbf{b}_5$	17.317	17.317	2078	3171	0.008	0.042	0.043	0.950	0.279
$\mathbf{b}_6$	17.528	17.528	2213	3041	0.015	0.041	0.044	0.951	0.274
$\mathbf{b}_7$	17.616	17.616	2312	2838	0.020	0.040	0.045	0.952	0.272
$\mathbf{b}_8$	17.424	17.424	2334	2539	0.021	0.040	0.046	0.950	0.278
$\mathbf{b}_9$	17.357	17.357	2359	2261	0.017	0.040	0.044	0.949	0.283
$\mathbf{b}_{10}$	17.274	17.274	2356	1973	0.009	0.040	0.041	0.951	0.288
$\mathbf{b}_{11}$	18.406	18.406	2551	1951	-0.000	0.040	0.040	0.954	0.285
$\mathbf{b}_{12}$	17.297	17.297	2359	2086	-0.009	0.042	0.043	0.953	0.294
$\mathbf{b}_{13}$	17.369	17.369	2333	2472	-0.015	0.042	0.045	0.955	0.293
$\mathbf{b}_{14}$	17.534	17.534	2280	2862	-0.016	0.043	0.046	0.958	0.291
$\mathbf{b}_{15}$	17.147	17.147	2080	3047	-0.009	0.045	0.046	0.952	0.302
$\mathbf{b}_{16}$	16.682	16.682	1869	3114	-0.001	0.046	0.046	0.955	0.315
$\mathbf{b}_{17}$	16.578	16.578	1771	3166	0.001	0.046	0.046	0.949	0.325
$\mathbf{b}_{18}$	16.978	16.978	1779	3293	0.000	0.045	0.045	0.951	0.327
$\mathbf{b}_{19}$	17.884	17.884	1843	3543	-0.002	0.045	0.045	0.949	0.323
$\mathbf{b}_{20}$	18.952	18.952	1897	3799	-0.003	0.046	0.046	0.946	0.319
$\mathbf{b}_{21}$	20.130	20.130	1928	4034	-0.005	0.045	0.046	0.951	0.318
`Uniform`								0.948	0.447
`WBATE`					0.001	0.018	0.018	0.951	0.101
`LBATE`					0.041	0.032	0.052	0.985	0.473

Table SA-29: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.887	17.887	1892	3622	-0.003	0.040	0.040	0.952	0.290
$\mathbf{b}_2$	16.808	16.808	1774	3244	-0.002	0.041	0.041	0.956	0.299
$\mathbf{b}_3$	15.591	15.591	1615	2807	-0.001	0.043	0.043	0.954	0.313
$\mathbf{b}_4$	13.963	13.963	1377	2248	0.000	0.047	0.047	0.945	0.342
$\mathbf{b}_5$	12.000	12.000	1082	1642	0.001	0.053	0.053	0.939	0.392
$\mathbf{b}_6$	10.000	10.000	788	1113	0.002	0.062	0.062	0.940	0.466
$\mathbf{b}_7$	8.168	8.168	543	715	0.001	0.077	0.077	0.941	0.568
$\mathbf{b}_8$	7.637	7.637	494	577	0.003	0.084	0.084	0.940	0.614
$\mathbf{b}_9$	7.604	7.604	526	487	0.010	0.086	0.086	0.940	0.627
$\mathbf{b}_{10}$	7.568	7.568	562	384	0.008	0.086	0.087	0.942	0.641
$\mathbf{b}_{11}$	8.064	8.064	672	322	0.001	0.085	0.085	0.937	0.638
$\mathbf{b}_{12}$	7.578	7.578	547	425	-0.005	0.088	0.089	0.940	0.651
$\mathbf{b}_{13}$	7.610	7.610	496	558	-0.003	0.086	0.086	0.944	0.644
$\mathbf{b}_{14}$	8.430	8.430	576	769	-0.001	0.078	0.078	0.942	0.582
$\mathbf{b}_{15}$	11.200	11.200	962	1418	-0.000	0.061	0.061	0.942	0.447
$\mathbf{b}_{16}$	13.926	13.926	1372	2236	0.000	0.050	0.050	0.947	0.369
$\mathbf{b}_{17}$	15.794	15.794	1641	2883	0.001	0.047	0.047	0.945	0.338
$\mathbf{b}_{18}$	16.820	16.820	1756	3230	-0.000	0.046	0.046	0.952	0.329
$\mathbf{b}_{19}$	17.866	17.866	1841	3536	-0.002	0.045	0.045	0.950	0.323
$\mathbf{b}_{20}$	18.951	18.951	1897	3799	-0.003	0.046	0.046	0.946	0.319
$\mathbf{b}_{21}$	20.130	20.130	1928	4034	-0.005	0.045	0.046	0.951	0.318
`Uniform`								0.921	0.694
`WBATE`					0.000	0.022	0.022	0.944	0.127
`LBATE`					0.099	0.057	0.115	0.973	0.863

Table SA-30: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.979	7.979	540	716	-0.003	0.084	0.084	0.950	0.327
$\mathbf{b}_2$	7.692	7.692	514	675	-0.001	0.084	0.084	0.954	0.331
$\mathbf{b}_3$	7.537	7.537	499	652	-0.000	0.084	0.084	0.955	0.332
$\mathbf{b}_4$	7.517	7.517	497	648	-0.000	0.086	0.086	0.944	0.328
$\mathbf{b}_5$	7.587	7.587	500	653	0.001	0.085	0.085	0.945	0.322
$\mathbf{b}_6$	7.679	7.679	501	656	0.001	0.083	0.083	0.949	0.317
$\mathbf{b}_7$	7.718	7.718	493	643	0.001	0.083	0.083	0.951	0.316
$\mathbf{b}_8$	7.634	7.634	494	576	0.003	0.084	0.084	0.951	0.321
$\mathbf{b}_9$	7.604	7.604	526	487	0.010	0.086	0.086	0.947	0.327
$\mathbf{b}_{10}$	7.568	7.568	562	384	0.008	0.086	0.087	0.949	0.332
$\mathbf{b}_{11}$	8.064	8.064	672	322	0.001	0.085	0.085	0.952	0.332
$\mathbf{b}_{12}$	7.578	7.578	547	425	-0.005	0.088	0.089	0.952	0.340
$\mathbf{b}_{13}$	7.609	7.609	496	558	-0.003	0.086	0.086	0.953	0.337
$\mathbf{b}_{14}$	7.682	7.682	490	642	-0.000	0.086	0.086	0.951	0.336
$\mathbf{b}_{15}$	7.512	7.512	488	636	0.000	0.091	0.091	0.942	0.346
$\mathbf{b}_{16}$	7.309	7.309	472	612	0.005	0.094	0.094	0.946	0.360
$\mathbf{b}_{17}$	7.263	7.263	466	603	0.004	0.098	0.098	0.943	0.370
$\mathbf{b}_{18}$	7.438	7.438	478	622	0.001	0.096	0.096	0.947	0.371
$\mathbf{b}_{19}$	7.835	7.835	508	670	-0.002	0.094	0.094	0.950	0.364
$\mathbf{b}_{20}$	8.303	8.303	537	720	-0.000	0.093	0.093	0.946	0.360
$\mathbf{b}_{21}$	8.819	8.819	560	766	0.000	0.092	0.092	0.947	0.357
`Uniform`								0.933	0.519
`WBATE`					0.001	0.027	0.027	0.949	0.104
`LBATE`					0.134	0.058	0.146	0.983	0.545

Table SA-31: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.349	17.349	1823	3432	-0.005	0.044	0.044	0.957	0.301
$\mathbf{b}_2$	16.879	16.879	1796	3288	-0.004	0.044	0.044	0.959	0.302
$\mathbf{b}_3$	16.600	16.600	1802	3173	-0.004	0.044	0.044	0.960	0.302
$\mathbf{b}_4$	16.606	16.606	1872	3104	-0.001	0.046	0.046	0.958	0.299
$\mathbf{b}_5$	16.893	16.893	2007	3068	0.005	0.048	0.048	0.951	0.293
$\mathbf{b}_6$	17.267	17.267	2163	3001	0.011	0.047	0.049	0.952	0.286
$\mathbf{b}_7$	17.265	17.265	2238	2782	0.016	0.047	0.050	0.952	0.285
$\mathbf{b}_8$	17.338	17.338	2311	2557	0.018	0.046	0.049	0.949	0.286
$\mathbf{b}_9$	17.379	17.379	2356	2307	0.015	0.044	0.047	0.956	0.289
$\mathbf{b}_{10}$	17.542	17.542	2399	2077	0.008	0.044	0.045	0.956	0.290
$\mathbf{b}_{11}$	18.603	18.603	2573	2042	-0.001	0.043	0.043	0.959	0.288
$\mathbf{b}_{12}$	17.562	17.562	2403	2187	-0.009	0.045	0.046	0.956	0.296
$\mathbf{b}_{13}$	17.605	17.605	2376	2569	-0.014	0.046	0.048	0.958	0.295
$\mathbf{b}_{14}$	17.447	17.447	2262	2877	-0.012	0.049	0.050	0.957	0.300
$\mathbf{b}_{15}$	16.875	16.875	2039	3004	-0.005	0.053	0.053	0.960	0.316
$\mathbf{b}_{16}$	16.081	16.081	1785	2944	0.003	0.052	0.052	0.959	0.335
$\mathbf{b}_{17}$	15.755	15.755	1653	2902	0.005	0.051	0.052	0.950	0.346
$\mathbf{b}_{18}$	15.961	15.961	1637	2941	0.003	0.051	0.051	0.953	0.349
$\mathbf{b}_{19}$	16.555	16.555	1669	3063	0.001	0.051	0.051	0.952	0.347
$\mathbf{b}_{20}$	17.249	17.249	1687	3182	0.000	0.052	0.052	0.951	0.349
$\mathbf{b}_{21}$	17.956	17.956	1676	3259	-0.000	0.053	0.053	0.954	0.355
`Uniform`								0.946	0.469
`WBATE`					0.002	0.019	0.019	0.948	0.104
`LBATE`					0.053	0.042	0.067	0.985	0.532

Table SA-32: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.534	19.534	2120	4260	-0.002	0.027	0.028	0.949	0.207
$\mathbf{b}_2$	19.152	19.152	2138	4114	-0.002	0.028	0.028	0.945	0.210
$\mathbf{b}_3$	18.902	18.902	2185	3961	-0.002	0.028	0.028	0.944	0.212
$\mathbf{b}_4$	18.805	18.805	2271	3805	-0.002	0.029	0.029	0.945	0.214
$\mathbf{b}_5$	18.832	18.832	2383	3639	-0.002	0.029	0.029	0.945	0.215
$\mathbf{b}_6$	19.003	19.003	2517	3478	-0.002	0.029	0.029	0.945	0.216
$\mathbf{b}_7$	19.115	19.115	2621	3265	-0.002	0.030	0.030	0.944	0.219
$\mathbf{b}_8$	19.116	19.116	2679	2996	-0.001	0.031	0.031	0.948	0.224
$\mathbf{b}_9$	19.218	19.218	2729	2738	0.000	0.032	0.032	0.945	0.232
$\mathbf{b}_{10}$	19.480	19.480	2777	2508	0.000	0.035	0.035	0.945	0.250
$\mathbf{b}_{11}$	21.345	21.345	3064	2673	-0.001	0.036	0.036	0.942	0.281
$\mathbf{b}_{12}$	19.319	19.319	2751	2587	-0.004	0.033	0.033	0.946	0.240
$\mathbf{b}_{13}$	19.199	19.199	2704	2962	-0.006	0.030	0.031	0.942	0.224
$\mathbf{b}_{14}$	19.204	19.204	2625	3342	-0.007	0.029	0.029	0.944	0.216
$\mathbf{b}_{15}$	18.993	18.993	2458	3608	-0.005	0.029	0.029	0.948	0.213
$\mathbf{b}_{16}$	18.625	18.625	2237	3746	-0.002	0.029	0.029	0.950	0.214
$\mathbf{b}_{17}$	18.421	18.421	2068	3821	0.000	0.029	0.029	0.948	0.215
$\mathbf{b}_{18}$	18.678	18.678	2014	3935	0.001	0.028	0.028	0.944	0.213
$\mathbf{b}_{19}$	19.490	19.490	2043	4167	0.002	0.027	0.027	0.948	0.206
$\mathbf{b}_{20}$	20.668	20.668	2098	4464	0.003	0.026	0.026	0.948	0.198
$\mathbf{b}_{21}$	21.765	21.765	2107	4658	0.004	0.026	0.026	0.949	0.193
`Uniform`								0.935	0.328
`WBATE`					-0.001	0.015	0.015	0.949	0.085
`LBATE`					0.011	0.024	0.026	0.993	0.324

Table SA-33: First-Stage Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.862	18.862	2022	3994	-0.002	0.028	0.028	0.948	0.213
$\mathbf{b}_2$	17.546	17.546	1877	3512	-0.001	0.030	0.030	0.948	0.226
$\mathbf{b}_3$	15.909	15.909	1661	2914	-0.001	0.033	0.033	0.943	0.248
$\mathbf{b}_4$	13.997	13.997	1382	2257	-0.000	0.038	0.038	0.942	0.282
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.001	0.044	0.044	0.940	0.331
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.001	0.053	0.053	0.948	0.401
$\mathbf{b}_7$	8.562	8.562	596	783	-0.002	0.062	0.063	0.941	0.477
$\mathbf{b}_8$	8.375	8.375	599	680	-0.002	0.068	0.068	0.935	0.501
$\mathbf{b}_9$	8.420	8.420	647	585	-0.000	0.070	0.070	0.932	0.519
$\mathbf{b}_{10}$	8.534	8.534	708	483	-0.000	0.075	0.075	0.931	0.539
$\mathbf{b}_{11}$	9.351	9.351	877	445	-0.001	0.088	0.088	0.919	0.659
$\mathbf{b}_{12}$	8.464	8.464	680	522	-0.004	0.072	0.072	0.933	0.531
$\mathbf{b}_{13}$	8.411	8.411	612	667	-0.003	0.068	0.068	0.935	0.501
$\mathbf{b}_{14}$	8.717	8.717	614	821	-0.002	0.062	0.062	0.940	0.463
$\mathbf{b}_{15}$	11.200	11.200	961	1419	-0.000	0.047	0.047	0.936	0.353
$\mathbf{b}_{16}$	13.996	13.996	1382	2258	-0.000	0.037	0.037	0.942	0.279
$\mathbf{b}_{17}$	16.483	16.483	1739	3123	0.000	0.031	0.031	0.949	0.237
$\mathbf{b}_{18}$	18.112	18.112	1930	3712	0.001	0.029	0.029	0.945	0.218
$\mathbf{b}_{19}$	19.440	19.440	2038	4146	0.002	0.027	0.027	0.948	0.207
$\mathbf{b}_{20}$	20.668	20.668	2098	4464	0.003	0.026	0.026	0.948	0.198
$\mathbf{b}_{21}$	21.765	21.765	2107	4658	0.004	0.026	0.026	0.949	0.193
`Uniform`								0.884	0.551
`WBATE`					-0.001	0.019	0.019	0.952	0.110
`LBATE`					0.034	0.035	0.049	0.979	0.637

Table SA-34: First-Stage Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.558	8.558	609	823	-0.001	0.064	0.064	0.948	0.244
$\mathbf{b}_2$	8.391	8.391	598	803	-0.001	0.064	0.064	0.946	0.247
$\mathbf{b}_3$	8.281	8.281	589	788	-0.001	0.065	0.065	0.943	0.250
$\mathbf{b}_4$	8.239	8.239	583	779	-0.000	0.066	0.066	0.939	0.251
$\mathbf{b}_5$	8.251	8.251	579	773	-0.001	0.066	0.066	0.941	0.251
$\mathbf{b}_6$	8.325	8.325	577	772	-0.002	0.065	0.065	0.946	0.251
$\mathbf{b}_7$	8.375	8.375	575	752	-0.003	0.065	0.065	0.950	0.253
$\mathbf{b}_8$	8.375	8.375	599	680	-0.002	0.068	0.068	0.943	0.260
$\mathbf{b}_9$	8.420	8.420	647	585	-0.000	0.070	0.070	0.942	0.269
$\mathbf{b}_{10}$	8.534	8.534	708	483	-0.000	0.075	0.075	0.941	0.289
$\mathbf{b}_{11}$	9.351	9.351	877	445	-0.001	0.088	0.088	0.940	0.339
$\mathbf{b}_{12}$	8.464	8.464	680	522	-0.004	0.072	0.072	0.944	0.278
$\mathbf{b}_{13}$	8.411	8.411	612	667	-0.003	0.068	0.068	0.946	0.260
$\mathbf{b}_{14}$	8.413	8.413	579	768	-0.001	0.065	0.065	0.939	0.249
$\mathbf{b}_{15}$	8.321	8.321	583	781	-0.000	0.065	0.065	0.939	0.247
$\mathbf{b}_{16}$	8.160	8.160	573	764	0.001	0.065	0.065	0.943	0.251
$\mathbf{b}_{17}$	8.070	8.070	561	747	0.000	0.065	0.065	0.948	0.254
$\mathbf{b}_{18}$	8.183	8.183	566	756	-0.001	0.066	0.066	0.939	0.252
$\mathbf{b}_{19}$	8.539	8.539	590	798	-0.002	0.063	0.063	0.944	0.244
$\mathbf{b}_{20}$	9.055	9.055	623	856	-0.000	0.060	0.060	0.946	0.234
$\mathbf{b}_{21}$	9.535	9.535	639	893	0.001	0.059	0.059	0.943	0.228
`Uniform`								0.921	0.391
`WBATE`					-0.001	0.022	0.022	0.955	0.087
`LBATE`					0.069	0.043	0.082	0.989	0.386

Table SA-35: First-Stage Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.325	19.325	2100	4182	-0.002	0.029	0.029	0.949	0.212
$\mathbf{b}_2$	19.171	19.171	2152	4123	-0.002	0.029	0.029	0.948	0.213
$\mathbf{b}_3$	19.127	19.127	2235	4046	-0.002	0.029	0.029	0.948	0.213
$\mathbf{b}_4$	19.198	19.198	2351	3949	-0.003	0.029	0.030	0.948	0.213
$\mathbf{b}_5$	19.404	19.404	2495	3845	-0.003	0.030	0.030	0.945	0.213
$\mathbf{b}_6$	19.638	19.638	2635	3703	-0.002	0.030	0.030	0.946	0.214
$\mathbf{b}_7$	19.901	19.901	2763	3534	-0.002	0.030	0.030	0.945	0.215
$\mathbf{b}_8$	20.219	20.219	2874	3351	-0.001	0.031	0.031	0.949	0.217
$\mathbf{b}_9$	20.701	20.701	2983	3189	0.000	0.031	0.031	0.949	0.220
$\mathbf{b}_{10}$	21.556	21.556	3119	3110	0.000	0.033	0.033	0.950	0.232
$\mathbf{b}_{11}$	23.400	23.400	3362	3287	-0.001	0.035	0.035	0.947	0.261
$\mathbf{b}_{12}$	21.216	21.216	3072	3143	-0.004	0.032	0.032	0.947	0.225
$\mathbf{b}_{13}$	20.499	20.499	2936	3369	-0.007	0.030	0.031	0.946	0.215
$\mathbf{b}_{14}$	20.118	20.118	2793	3651	-0.007	0.029	0.030	0.945	0.211
$\mathbf{b}_{15}$	19.695	19.695	2591	3862	-0.005	0.030	0.030	0.946	0.210
$\mathbf{b}_{16}$	19.024	19.024	2321	3897	-0.001	0.030	0.030	0.951	0.215
$\mathbf{b}_{17}$	18.414	18.414	2081	3827	0.001	0.030	0.030	0.951	0.219
$\mathbf{b}_{18}$	18.243	18.243	1959	3776	0.001	0.030	0.030	0.943	0.220
$\mathbf{b}_{19}$	18.530	18.530	1924	3797	0.002	0.030	0.030	0.950	0.218
$\mathbf{b}_{20}$	19.104	19.104	1912	3865	0.002	0.030	0.030	0.948	0.216
$\mathbf{b}_{21}$	19.617	19.617	1867	3854	0.002	0.030	0.030	0.952	0.217
`Uniform`								0.935	0.326
`WBATE`					-0.002	0.015	0.015	0.948	0.084
`LBATE`					0.013	0.027	0.030	0.991	0.343

Table SA-36: First-Stage Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.822	19.822	2159	4374	-0.001	0.027	0.027	0.949	0.204
$\mathbf{b}_2$	19.210	19.210	2146	4137	-0.001	0.027	0.028	0.951	0.209
$\mathbf{b}_3$	18.737	18.737	2155	3903	-0.001	0.028	0.028	0.950	0.214
$\mathbf{b}_4$	18.364	18.364	2185	3658	-0.001	0.029	0.029	0.949	0.218
$\mathbf{b}_5$	18.091	18.091	2234	3405	-0.002	0.030	0.030	0.948	0.223
$\mathbf{b}_6$	17.917	17.917	2293	3150	-0.002	0.031	0.031	0.942	0.229
$\mathbf{b}_7$	17.759	17.759	2342	2876	-0.001	0.032	0.032	0.945	0.235
$\mathbf{b}_8$	17.432	17.432	2336	2540	-0.001	0.034	0.034	0.945	0.246
$\mathbf{b}_9$	17.290	17.290	2346	2244	0.000	0.036	0.036	0.942	0.257
$\mathbf{b}_{10}$	17.153	17.153	2333	1945	0.000	0.040	0.040	0.939	0.281
$\mathbf{b}_{11}$	18.116	18.116	2499	1885	-0.000	0.044	0.044	0.942	0.334
$\mathbf{b}_{12}$	17.198	17.198	2339	2062	-0.003	0.038	0.038	0.943	0.268
$\mathbf{b}_{13}$	17.364	17.364	2332	2471	-0.005	0.033	0.034	0.946	0.247
$\mathbf{b}_{14}$	17.720	17.720	2319	2912	-0.005	0.031	0.032	0.945	0.233
$\mathbf{b}_{15}$	17.870	17.870	2228	3261	-0.004	0.031	0.031	0.941	0.225
$\mathbf{b}_{16}$	18.113	18.113	2137	3576	-0.001	0.030	0.030	0.939	0.219
$\mathbf{b}_{17}$	18.540	18.540	2085	3865	0.001	0.029	0.029	0.945	0.213
$\mathbf{b}_{18}$	19.281	19.281	2097	4170	0.002	0.028	0.028	0.948	0.206
$\mathbf{b}_{19}$	20.431	20.431	2159	4545	0.003	0.026	0.026	0.950	0.196
$\mathbf{b}_{20}$	21.957	21.957	2239	4992	0.004	0.025	0.025	0.945	0.186
$\mathbf{b}_{21}$	23.561	23.561	2283	5391	0.005	0.024	0.024	0.948	0.178
`Uniform`								0.928	0.344
`WBATE`					-0.001	0.015	0.016	0.950	0.087
`LBATE`					0.011	0.023	0.025	0.992	0.336

Table SA-37: First-Stage Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.027	19.027	2044	4060	-0.001	0.028	0.028	0.951	0.211
$\mathbf{b}_2$	17.579	17.579	1882	3526	-0.001	0.029	0.029	0.951	0.226
$\mathbf{b}_3$	15.902	15.902	1660	2912	-0.000	0.032	0.032	0.952	0.248
$\mathbf{b}_4$	13.995	13.995	1382	2257	-0.000	0.037	0.037	0.946	0.282
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.000	0.043	0.043	0.945	0.331
$\mathbf{b}_6$	10.000	10.000	788	1112	-0.001	0.052	0.052	0.941	0.401
$\mathbf{b}_7$	8.181	8.181	545	717	-0.000	0.066	0.066	0.941	0.498
$\mathbf{b}_8$	7.642	7.642	495	577	-0.000	0.073	0.073	0.936	0.550
$\mathbf{b}_9$	7.575	7.575	521	483	-0.002	0.077	0.077	0.936	0.574
$\mathbf{b}_{10}$	7.515	7.515	554	378	-0.002	0.085	0.085	0.925	0.611
$\mathbf{b}_{11}$	7.937	7.937	653	311	-0.001	0.107	0.107	0.908	0.777
$\mathbf{b}_{12}$	7.534	7.534	540	421	-0.003	0.080	0.080	0.927	0.593
$\mathbf{b}_{13}$	7.608	7.608	496	557	-0.000	0.073	0.073	0.935	0.553
$\mathbf{b}_{14}$	8.434	8.434	576	769	0.001	0.063	0.063	0.934	0.477
$\mathbf{b}_{15}$	11.200	11.200	962	1419	0.000	0.047	0.047	0.940	0.353
$\mathbf{b}_{16}$	13.991	13.991	1382	2256	0.000	0.037	0.037	0.941	0.279
$\mathbf{b}_{17}$	16.545	16.545	1748	3147	0.001	0.032	0.032	0.938	0.236
$\mathbf{b}_{18}$	18.559	18.559	1990	3886	0.001	0.028	0.028	0.947	0.212
$\mathbf{b}_{19}$	20.288	20.288	2142	4486	0.002	0.026	0.027	0.950	0.197
$\mathbf{b}_{20}$	21.947	21.947	2238	4988	0.004	0.025	0.025	0.945	0.186
$\mathbf{b}_{21}$	23.560	23.560	2283	5391	0.005	0.024	0.024	0.948	0.178
`Uniform`								0.873	0.580
`WBATE`					0.000	0.020	0.020	0.946	0.113
`LBATE`					0.043	0.042	0.060	0.973	0.721

Table SA-38: First-Stage Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.684	8.684	625	848	-0.001	0.062	0.061	0.949	0.241
$\mathbf{b}_2$	8.416	8.416	601	807	-0.000	0.062	0.062	0.953	0.247
$\mathbf{b}_3$	8.209	8.209	580	773	-0.000	0.065	0.065	0.947	0.252
$\mathbf{b}_4$	8.045	8.045	559	740	-0.000	0.067	0.067	0.941	0.256
$\mathbf{b}_5$	7.926	7.926	539	711	-0.001	0.068	0.068	0.944	0.261
$\mathbf{b}_6$	7.850	7.850	521	684	-0.001	0.068	0.068	0.946	0.266
$\mathbf{b}_7$	7.780	7.780	501	651	-0.000	0.070	0.070	0.942	0.272
$\mathbf{b}_8$	7.637	7.637	495	576	-0.000	0.073	0.073	0.948	0.285
$\mathbf{b}_9$	7.575	7.575	521	483	-0.002	0.077	0.077	0.945	0.298
$\mathbf{b}_{10}$	7.515	7.515	554	378	-0.002	0.085	0.085	0.937	0.325
$\mathbf{b}_{11}$	7.937	7.937	653	311	-0.001	0.107	0.107	0.930	0.402
$\mathbf{b}_{12}$	7.534	7.534	540	421	-0.003	0.080	0.080	0.943	0.310
$\mathbf{b}_{13}$	7.607	7.607	496	557	-0.000	0.073	0.073	0.946	0.287
$\mathbf{b}_{14}$	7.763	7.763	500	654	0.001	0.070	0.070	0.943	0.270
$\mathbf{b}_{15}$	7.829	7.829	524	689	0.000	0.069	0.069	0.942	0.263
$\mathbf{b}_{16}$	7.935	7.935	546	720	0.001	0.067	0.067	0.944	0.257
$\mathbf{b}_{17}$	8.122	8.122	568	755	0.002	0.066	0.066	0.942	0.251
$\mathbf{b}_{18}$	8.447	8.447	598	805	0.000	0.063	0.063	0.944	0.243
$\mathbf{b}_{19}$	8.951	8.951	640	877	0.001	0.059	0.059	0.949	0.232
$\mathbf{b}_{20}$	9.620	9.620	691	969	0.002	0.057	0.057	0.944	0.220
$\mathbf{b}_{21}$	10.322	10.322	731	1051	0.002	0.055	0.055	0.942	0.210
`Uniform`								0.910	0.410
`WBATE`					-0.000	0.023	0.023	0.947	0.089
`LBATE`					0.074	0.044	0.086	0.985	0.405

Table SA-39: First-Stage Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.920	18.920	2042	4036	-0.002	0.029	0.029	0.947	0.217
$\mathbf{b}_2$	18.614	18.614	2065	3924	-0.002	0.029	0.029	0.952	0.220
$\mathbf{b}_3$	18.397	18.397	2110	3794	-0.002	0.030	0.030	0.949	0.222
$\mathbf{b}_4$	18.284	18.284	2182	3647	-0.002	0.031	0.031	0.949	0.224
$\mathbf{b}_5$	18.172	18.172	2256	3454	-0.002	0.031	0.031	0.950	0.227
$\mathbf{b}_6$	18.104	18.104	2332	3236	-0.002	0.032	0.032	0.945	0.232
$\mathbf{b}_7$	17.715	17.715	2330	2898	-0.002	0.034	0.034	0.945	0.241
$\mathbf{b}_8$	17.474	17.474	2340	2587	-0.001	0.035	0.035	0.947	0.251
$\mathbf{b}_9$	17.306	17.306	2344	2288	0.000	0.038	0.038	0.946	0.263
$\mathbf{b}_{10}$	17.385	17.385	2370	2040	0.000	0.041	0.041	0.939	0.284
$\mathbf{b}_{11}$	18.282	18.282	2516	1968	-0.000	0.046	0.046	0.943	0.339
$\mathbf{b}_{12}$	17.522	17.522	2395	2180	-0.003	0.038	0.039	0.943	0.269
$\mathbf{b}_{13}$	17.903	17.903	2435	2652	-0.005	0.034	0.034	0.950	0.246
$\mathbf{b}_{14}$	18.374	18.374	2449	3140	-0.005	0.032	0.032	0.945	0.231
$\mathbf{b}_{15}$	18.805	18.805	2421	3586	-0.004	0.031	0.031	0.941	0.221
$\mathbf{b}_{16}$	18.919	18.919	2307	3867	-0.001	0.031	0.031	0.941	0.217
$\mathbf{b}_{17}$	18.942	18.942	2175	4021	0.001	0.030	0.030	0.944	0.214
$\mathbf{b}_{18}$	19.136	19.136	2090	4131	0.001	0.029	0.029	0.946	0.212
$\mathbf{b}_{19}$	19.695	19.695	2068	4278	0.003	0.028	0.028	0.948	0.208
$\mathbf{b}_{20}$	20.557	20.557	2072	4464	0.003	0.027	0.028	0.948	0.204
$\mathbf{b}_{21}$	21.470	21.470	2054	4601	0.004	0.028	0.028	0.947	0.201
`Uniform`								0.926	0.354
`WBATE`					-0.001	0.016	0.016	0.945	0.088
`LBATE`					0.014	0.026	0.029	0.991	0.366

Table SA-40: First-Stage Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.945	17.945	1903	3648	-0.005	0.030	0.030	0.947	0.226
$\mathbf{b}_2$	17.380	17.380	1868	3459	-0.004	0.030	0.031	0.950	0.232
$\mathbf{b}_3$	17.202	17.202	1891	3370	-0.004	0.031	0.031	0.947	0.234
$\mathbf{b}_4$	17.380	17.380	1997	3341	-0.002	0.031	0.031	0.947	0.232
$\mathbf{b}_5$	17.857	17.857	2186	3339	0.000	0.031	0.031	0.947	0.227
$\mathbf{b}_6$	18.430	18.430	2397	3310	0.003	0.030	0.030	0.946	0.223
$\mathbf{b}_7$	18.889	18.889	2574	3202	0.006	0.030	0.030	0.947	0.221
$\mathbf{b}_8$	18.988	18.988	2653	2961	0.006	0.030	0.031	0.947	0.225
$\mathbf{b}_9$	19.108	19.108	2707	2710	0.005	0.032	0.032	0.944	0.231
$\mathbf{b}_{10}$	19.393	19.393	2761	2487	0.001	0.034	0.034	0.947	0.248
$\mathbf{b}_{11}$	21.303	21.303	3057	2662	-0.006	0.036	0.037	0.947	0.278
$\mathbf{b}_{12}$	19.219	19.219	2732	2561	-0.007	0.033	0.034	0.949	0.238
$\mathbf{b}_{13}$	19.004	19.004	2665	2911	-0.007	0.030	0.031	0.943	0.223
$\mathbf{b}_{14}$	18.692	18.692	2519	3196	-0.006	0.030	0.030	0.939	0.219
$\mathbf{b}_{15}$	17.796	17.796	2212	3246	-0.003	0.030	0.031	0.947	0.224
$\mathbf{b}_{16}$	16.806	16.806	1891	3154	-0.002	0.031	0.031	0.943	0.234
$\mathbf{b}_{17}$	16.262	16.262	1722	3052	-0.001	0.032	0.032	0.946	0.240
$\mathbf{b}_{18}$	16.380	16.380	1697	3070	-0.000	0.031	0.031	0.944	0.239
$\mathbf{b}_{19}$	17.047	17.047	1735	3229	-0.000	0.031	0.031	0.943	0.233
$\mathbf{b}_{20}$	17.934	17.934	1774	3415	0.000	0.030	0.030	0.950	0.226
$\mathbf{b}_{21}$	18.901	18.901	1791	3574	0.000	0.029	0.029	0.947	0.221
`Uniform`								0.924	0.348
`WBATE`					-0.001	0.015	0.015	0.947	0.087
`LBATE`					0.017	0.026	0.031	0.985	0.354

Table SA-41: First-Stage Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.655	17.655	1862	3533	-0.005	0.030	0.030	0.946	0.229
$\mathbf{b}_2$	16.710	16.710	1760	3207	-0.004	0.031	0.031	0.951	0.239
$\mathbf{b}_3$	15.625	15.625	1619	2818	-0.003	0.033	0.033	0.947	0.253
$\mathbf{b}_4$	13.979	13.979	1379	2252	-0.002	0.037	0.037	0.944	0.283
$\mathbf{b}_5$	12.000	12.000	1080	1642	-0.001	0.044	0.044	0.942	0.331
$\mathbf{b}_6$	10.000	10.000	787	1114	-0.000	0.053	0.053	0.943	0.400
$\mathbf{b}_7$	8.509	8.509	589	775	-0.000	0.064	0.064	0.935	0.477
$\mathbf{b}_8$	8.319	8.319	591	673	0.001	0.067	0.067	0.938	0.503
$\mathbf{b}_9$	8.372	8.372	639	580	0.004	0.068	0.068	0.932	0.516
$\mathbf{b}_{10}$	8.496	8.496	702	480	0.003	0.073	0.073	0.933	0.536
$\mathbf{b}_{11}$	9.333	9.333	873	444	-0.002	0.086	0.086	0.918	0.652
$\mathbf{b}_{12}$	8.420	8.420	673	518	-0.002	0.070	0.070	0.933	0.525
$\mathbf{b}_{13}$	8.326	8.326	600	655	-0.000	0.067	0.067	0.931	0.498
$\mathbf{b}_{14}$	8.643	8.643	604	807	0.000	0.062	0.062	0.935	0.460
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.046	0.046	0.945	0.348
$\mathbf{b}_{16}$	13.958	13.958	1377	2246	-0.002	0.037	0.037	0.940	0.276
$\mathbf{b}_{17}$	15.667	15.667	1623	2837	-0.001	0.033	0.033	0.945	0.247
$\mathbf{b}_{18}$	16.274	16.274	1681	3028	-0.001	0.032	0.032	0.944	0.240
$\mathbf{b}_{19}$	17.039	17.039	1734	3226	-0.000	0.031	0.031	0.943	0.233
$\mathbf{b}_{20}$	17.934	17.934	1774	3415	0.000	0.030	0.030	0.950	0.226
$\mathbf{b}_{21}$	18.901	18.901	1791	3574	0.000	0.029	0.029	0.947	0.221
`Uniform`								0.882	0.560
`WBATE`					-0.001	0.019	0.019	0.947	0.110
`LBATE`					0.044	0.038	0.059	0.967	0.646

Table SA-42: First-Stage Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.862	7.862	525	695	0.000	0.069	0.069	0.943	0.268
$\mathbf{b}_2$	7.614	7.614	504	661	0.000	0.071	0.071	0.944	0.274
$\mathbf{b}_3$	7.536	7.536	498	652	-0.000	0.072	0.072	0.942	0.276
$\mathbf{b}_4$	7.614	7.614	507	666	-0.001	0.070	0.070	0.946	0.273
$\mathbf{b}_5$	7.823	7.823	527	697	-0.000	0.069	0.069	0.944	0.266
$\mathbf{b}_6$	8.074	8.074	547	729	-0.000	0.067	0.067	0.944	0.259
$\mathbf{b}_7$	8.275	8.275	563	736	-0.000	0.067	0.067	0.944	0.255
$\mathbf{b}_8$	8.319	8.319	591	673	0.001	0.067	0.067	0.942	0.260
$\mathbf{b}_9$	8.372	8.372	639	580	0.004	0.068	0.068	0.942	0.268
$\mathbf{b}_{10}$	8.496	8.496	702	480	0.003	0.073	0.073	0.945	0.287
$\mathbf{b}_{11}$	9.333	9.333	873	444	-0.002	0.086	0.086	0.943	0.335
$\mathbf{b}_{12}$	8.420	8.420	673	518	-0.002	0.070	0.070	0.947	0.275
$\mathbf{b}_{13}$	8.326	8.326	600	655	-0.000	0.067	0.067	0.944	0.259
$\mathbf{b}_{14}$	8.189	8.189	553	729	0.000	0.067	0.067	0.940	0.253
$\mathbf{b}_{15}$	7.797	7.797	522	687	-0.000	0.068	0.068	0.945	0.262
$\mathbf{b}_{16}$	7.363	7.363	479	620	-0.000	0.072	0.072	0.934	0.275
$\mathbf{b}_{17}$	7.124	7.124	450	579	-0.001	0.073	0.073	0.942	0.283
$\mathbf{b}_{18}$	7.176	7.176	448	577	0.001	0.074	0.074	0.938	0.283
$\mathbf{b}_{19}$	7.468	7.468	466	607	0.003	0.071	0.071	0.941	0.275
$\mathbf{b}_{20}$	7.857	7.857	486	643	0.001	0.070	0.070	0.945	0.267
$\mathbf{b}_{21}$	8.281	8.281	501	672	-0.000	0.067	0.067	0.942	0.261
`Uniform`								0.909	0.415
`WBATE`					0.000	0.023	0.023	0.947	0.089
`LBATE`					0.086	0.047	0.099	0.979	0.414

Table SA-43: First-Stage Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.303	17.303	1816	3409	-0.005	0.031	0.032	0.946	0.235
$\mathbf{b}_2$	16.910	16.910	1799	3295	-0.005	0.032	0.032	0.952	0.240
$\mathbf{b}_3$	16.825	16.825	1837	3247	-0.004	0.032	0.032	0.950	0.242
$\mathbf{b}_4$	17.058	17.058	1950	3248	-0.003	0.032	0.033	0.948	0.240
$\mathbf{b}_5$	17.578	17.578	2135	3278	-0.000	0.032	0.032	0.949	0.236
$\mathbf{b}_6$	18.313	18.313	2368	3312	0.003	0.032	0.032	0.945	0.230
$\mathbf{b}_7$	19.002	19.002	2581	3280	0.005	0.032	0.032	0.944	0.226
$\mathbf{b}_8$	19.652	19.652	2762	3196	0.006	0.031	0.031	0.945	0.223
$\mathbf{b}_9$	20.340	20.340	2915	3094	0.004	0.031	0.032	0.943	0.223
$\mathbf{b}_{10}$	21.331	21.331	3079	3051	-0.000	0.033	0.033	0.948	0.232
$\mathbf{b}_{11}$	23.114	23.114	3319	3205	-0.007	0.035	0.036	0.947	0.261
$\mathbf{b}_{12}$	20.904	20.904	3020	3053	-0.009	0.032	0.033	0.949	0.225
$\mathbf{b}_{13}$	19.854	19.854	2815	3187	-0.008	0.031	0.032	0.942	0.218
$\mathbf{b}_{14}$	18.695	18.695	2510	3236	-0.006	0.032	0.032	0.939	0.225
$\mathbf{b}_{15}$	17.220	17.220	2101	3104	-0.003	0.033	0.033	0.945	0.238
$\mathbf{b}_{16}$	15.920	15.920	1748	2886	-0.001	0.034	0.034	0.941	0.251
$\mathbf{b}_{17}$	15.326	15.326	1584	2741	-0.000	0.034	0.034	0.942	0.257
$\mathbf{b}_{18}$	15.400	15.400	1560	2726	0.000	0.034	0.034	0.943	0.255
$\mathbf{b}_{19}$	15.907	15.907	1584	2816	0.000	0.033	0.033	0.943	0.249
$\mathbf{b}_{20}$	16.537	16.537	1600	2910	-0.000	0.033	0.033	0.950	0.245
$\mathbf{b}_{21}$	17.049	17.049	1573	2921	-0.001	0.033	0.033	0.948	0.245
`Uniform`								0.926	0.357
`WBATE`					-0.002	0.016	0.016	0.944	0.087
`LBATE`					0.021	0.029	0.036	0.984	0.380

Table SA-44: First-Stage Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.213	18.213	1939	3752	-0.005	0.029	0.030	0.951	0.223
$\mathbf{b}_2$	17.557	17.557	1894	3525	-0.005	0.030	0.030	0.947	0.230
$\mathbf{b}_3$	17.203	17.203	1893	3370	-0.004	0.030	0.030	0.950	0.234
$\mathbf{b}_4$	17.159	17.159	1958	3268	-0.002	0.031	0.031	0.949	0.235
$\mathbf{b}_5$	17.317	17.317	2078	3171	0.000	0.031	0.031	0.945	0.234
$\mathbf{b}_6$	17.528	17.528	2213	3041	0.003	0.031	0.031	0.943	0.234
$\mathbf{b}_7$	17.616	17.616	2312	2838	0.006	0.031	0.031	0.948	0.236
$\mathbf{b}_8$	17.424	17.424	2334	2539	0.007	0.032	0.033	0.944	0.244
$\mathbf{b}_9$	17.357	17.357	2359	2261	0.006	0.035	0.035	0.938	0.253
$\mathbf{b}_{10}$	17.274	17.274	2356	1973	0.002	0.038	0.038	0.941	0.276
$\mathbf{b}_{11}$	18.406	18.406	2551	1951	-0.004	0.042	0.042	0.944	0.325
$\mathbf{b}_{12}$	17.297	17.297	2359	2086	-0.007	0.037	0.037	0.946	0.263
$\mathbf{b}_{13}$	17.369	17.369	2333	2472	-0.007	0.032	0.033	0.947	0.244
$\mathbf{b}_{14}$	17.534	17.534	2280	2862	-0.006	0.031	0.031	0.951	0.233
$\mathbf{b}_{15}$	17.147	17.147	2080	3047	-0.004	0.031	0.031	0.946	0.232
$\mathbf{b}_{16}$	16.682	16.682	1869	3114	-0.002	0.031	0.031	0.947	0.235
$\mathbf{b}_{17}$	16.578	16.578	1771	3166	-0.002	0.031	0.031	0.952	0.236
$\mathbf{b}_{18}$	16.978	16.978	1779	3293	-0.002	0.030	0.030	0.950	0.231
$\mathbf{b}_{19}$	17.884	17.884	1843	3543	-0.001	0.029	0.029	0.948	0.223
$\mathbf{b}_{20}$	18.952	18.952	1897	3799	-0.001	0.028	0.028	0.942	0.214
$\mathbf{b}_{21}$	20.130	20.130	1928	4034	0.000	0.027	0.027	0.946	0.208
`Uniform`								0.935	0.361
`WBATE`					-0.001	0.015	0.015	0.948	0.088
`LBATE`					0.016	0.025	0.029	0.990	0.363

Table SA-45: First-Stage Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.887	17.887	1892	3622	-0.005	0.030	0.030	0.952	0.226
$\mathbf{b}_2$	16.808	16.808	1774	3244	-0.004	0.031	0.031	0.948	0.238
$\mathbf{b}_3$	15.591	15.591	1615	2807	-0.003	0.033	0.033	0.952	0.254
$\mathbf{b}_4$	13.963	13.963	1377	2248	-0.002	0.037	0.037	0.946	0.283
$\mathbf{b}_5$	12.000	12.000	1082	1642	-0.000	0.043	0.043	0.943	0.331
$\mathbf{b}_6$	10.000	10.000	788	1113	0.001	0.053	0.053	0.939	0.400
$\mathbf{b}_7$	8.168	8.168	543	715	0.001	0.066	0.066	0.935	0.495
$\mathbf{b}_8$	7.637	7.637	494	577	0.003	0.074	0.074	0.923	0.545
$\mathbf{b}_9$	7.604	7.604	526	487	0.006	0.078	0.078	0.925	0.566
$\mathbf{b}_{10}$	7.568	7.568	562	384	0.005	0.083	0.083	0.925	0.598
$\mathbf{b}_{11}$	8.064	8.064	672	322	-0.000	0.102	0.102	0.913	0.756
$\mathbf{b}_{12}$	7.578	7.578	547	425	-0.002	0.079	0.079	0.925	0.582
$\mathbf{b}_{13}$	7.610	7.610	496	558	0.000	0.073	0.073	0.929	0.543
$\mathbf{b}_{14}$	8.430	8.430	576	769	-0.000	0.062	0.062	0.932	0.471
$\mathbf{b}_{15}$	11.200	11.200	962	1418	-0.001	0.046	0.046	0.936	0.348
$\mathbf{b}_{16}$	13.926	13.926	1372	2236	-0.001	0.036	0.036	0.943	0.277
$\mathbf{b}_{17}$	15.794	15.794	1641	2883	-0.002	0.032	0.032	0.951	0.245
$\mathbf{b}_{18}$	16.820	16.820	1756	3230	-0.002	0.030	0.030	0.950	0.233
$\mathbf{b}_{19}$	17.866	17.866	1841	3536	-0.001	0.029	0.029	0.948	0.223
$\mathbf{b}_{20}$	18.951	18.951	1897	3799	-0.001	0.028	0.028	0.942	0.214
$\mathbf{b}_{21}$	20.130	20.130	1928	4034	0.000	0.027	0.027	0.946	0.208
`Uniform`								0.872	0.585
`WBATE`					-0.000	0.020	0.020	0.942	0.113
`LBATE`					0.053	0.046	0.070	0.963	0.718

Table SA-46: First-Stage Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.979	7.979	540	716	-0.003	0.069	0.069	0.942	0.264
$\mathbf{b}_2$	7.692	7.692	514	675	-0.002	0.070	0.070	0.945	0.272
$\mathbf{b}_3$	7.537	7.537	499	652	-0.002	0.070	0.070	0.952	0.276
$\mathbf{b}_4$	7.517	7.517	497	648	-0.000	0.071	0.071	0.945	0.276
$\mathbf{b}_5$	7.587	7.587	500	653	0.002	0.071	0.071	0.944	0.274
$\mathbf{b}_6$	7.679	7.679	501	656	0.002	0.070	0.070	0.945	0.272
$\mathbf{b}_7$	7.718	7.718	493	643	0.001	0.071	0.071	0.944	0.273
$\mathbf{b}_8$	7.634	7.634	494	576	0.003	0.074	0.075	0.936	0.283
$\mathbf{b}_9$	7.604	7.604	526	487	0.006	0.078	0.078	0.938	0.295
$\mathbf{b}_{10}$	7.568	7.568	562	384	0.005	0.083	0.083	0.936	0.319
$\mathbf{b}_{11}$	8.064	8.064	672	322	-0.000	0.102	0.102	0.942	0.390
$\mathbf{b}_{12}$	7.578	7.578	547	425	-0.002	0.079	0.079	0.940	0.304
$\mathbf{b}_{13}$	7.609	7.609	496	558	0.000	0.073	0.073	0.942	0.283
$\mathbf{b}_{14}$	7.682	7.682	490	642	0.000	0.069	0.069	0.947	0.269
$\mathbf{b}_{15}$	7.512	7.512	488	636	0.001	0.071	0.071	0.939	0.272
$\mathbf{b}_{16}$	7.309	7.309	472	612	0.002	0.072	0.072	0.940	0.277
$\mathbf{b}_{17}$	7.263	7.263	466	603	-0.001	0.072	0.072	0.947	0.278
$\mathbf{b}_{18}$	7.438	7.438	478	622	-0.001	0.070	0.070	0.948	0.273
$\mathbf{b}_{19}$	7.835	7.835	508	670	-0.001	0.068	0.068	0.945	0.263
$\mathbf{b}_{20}$	8.303	8.303	537	720	-0.001	0.067	0.067	0.943	0.253
$\mathbf{b}_{21}$	8.819	8.819	560	766	-0.001	0.063	0.063	0.944	0.245
`Uniform`								0.911	0.430
`WBATE`					0.000	0.023	0.023	0.948	0.090
`LBATE`					0.088	0.048	0.100	0.977	0.427

Table SA-47: First-Stage Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.349	17.349	1823	3432	-0.006	0.031	0.032	0.952	0.236
$\mathbf{b}_2$	16.879	16.879	1796	3288	-0.005	0.032	0.032	0.949	0.241
$\mathbf{b}_3$	16.600	16.600	1802	3173	-0.004	0.032	0.032	0.956	0.246
$\mathbf{b}_4$	16.606	16.606	1872	3104	-0.003	0.033	0.033	0.949	0.248
$\mathbf{b}_5$	16.893	16.893	2007	3068	-0.001	0.034	0.034	0.946	0.246
$\mathbf{b}_6$	17.267	17.267	2163	3001	0.002	0.034	0.034	0.946	0.244
$\mathbf{b}_7$	17.265	17.265	2238	2782	0.005	0.034	0.034	0.951	0.248
$\mathbf{b}_8$	17.338	17.338	2311	2557	0.006	0.035	0.035	0.947	0.252
$\mathbf{b}_9$	17.379	17.379	2356	2307	0.005	0.036	0.037	0.940	0.260
$\mathbf{b}_{10}$	17.542	17.542	2399	2077	0.002	0.039	0.039	0.944	0.278
$\mathbf{b}_{11}$	18.603	18.603	2573	2042	-0.004	0.045	0.045	0.946	0.329
$\mathbf{b}_{12}$	17.562	17.562	2403	2187	-0.007	0.038	0.038	0.948	0.265
$\mathbf{b}_{13}$	17.605	17.605	2376	2569	-0.007	0.034	0.034	0.949	0.246
$\mathbf{b}_{14}$	17.447	17.447	2262	2877	-0.005	0.033	0.033	0.948	0.241
$\mathbf{b}_{15}$	16.875	16.875	2039	3004	-0.003	0.033	0.034	0.946	0.244
$\mathbf{b}_{16}$	16.081	16.081	1785	2944	-0.001	0.034	0.034	0.945	0.251
$\mathbf{b}_{17}$	15.755	15.755	1653	2902	-0.001	0.033	0.033	0.950	0.253
$\mathbf{b}_{18}$	15.961	15.961	1637	2941	-0.001	0.032	0.032	0.948	0.248
$\mathbf{b}_{19}$	16.555	16.555	1669	3063	-0.001	0.032	0.032	0.949	0.242
$\mathbf{b}_{20}$	17.249	17.249	1687	3182	-0.001	0.032	0.032	0.942	0.237
$\mathbf{b}_{21}$	17.956	17.956	1676	3259	-0.001	0.032	0.032	0.945	0.235
`Uniform`								0.929	0.380
`WBATE`					-0.001	0.016	0.016	0.945	0.090
`LBATE`					0.021	0.029	0.036	0.986	0.400

Table SA-48: First-Stage Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.534	19.534	2120	4260	0.003	0.055	0.055	0.960	0.397
$\mathbf{b}_2$	19.152	19.152	2138	4114	0.003	0.056	0.056	0.956	0.400
$\mathbf{b}_3$	18.902	18.902	2185	3961	0.004	0.057	0.057	0.959	0.403
$\mathbf{b}_4$	18.805	18.805	2271	3805	0.007	0.058	0.058	0.957	0.404
$\mathbf{b}_5$	18.832	18.832	2383	3639	0.011	0.058	0.059	0.960	0.403
$\mathbf{b}_6$	19.003	19.003	2517	3478	0.015	0.057	0.059	0.958	0.402
$\mathbf{b}_7$	19.115	19.115	2621	3265	0.017	0.057	0.059	0.957	0.406
$\mathbf{b}_8$	19.116	19.116	2679	2996	0.016	0.058	0.060	0.956	0.412
$\mathbf{b}_9$	19.218	19.218	2729	2738	0.012	0.059	0.060	0.957	0.418
$\mathbf{b}_{10}$	19.480	19.480	2777	2508	0.007	0.060	0.060	0.965	0.427
$\mathbf{b}_{11}$	21.345	21.345	3064	2673	0.002	0.059	0.060	0.965	0.436
$\mathbf{b}_{12}$	19.319	19.319	2751	2587	-0.005	0.059	0.059	0.961	0.419
$\mathbf{b}_{13}$	19.199	19.199	2704	2962	-0.012	0.058	0.059	0.957	0.400
$\mathbf{b}_{14}$	19.204	19.204	2625	3342	-0.014	0.055	0.057	0.954	0.381
$\mathbf{b}_{15}$	18.993	18.993	2458	3608	-0.010	0.054	0.055	0.956	0.372
$\mathbf{b}_{16}$	18.625	18.625	2237	3746	-0.003	0.054	0.054	0.961	0.373
$\mathbf{b}_{17}$	18.421	18.421	2068	3821	0.001	0.052	0.052	0.963	0.375
$\mathbf{b}_{18}$	18.678	18.678	2014	3935	0.001	0.049	0.049	0.959	0.370
$\mathbf{b}_{19}$	19.490	19.490	2043	4167	0.000	0.048	0.048	0.952	0.356
$\mathbf{b}_{20}$	20.668	20.668	2098	4464	-0.002	0.047	0.047	0.954	0.340
$\mathbf{b}_{21}$	21.765	21.765	2107	4658	-0.004	0.046	0.046	0.951	0.329
`Uniform`								0.970	0.587
`WBATE`					0.002	0.026	0.026	0.952	0.146
`LBATE`					0.053	0.044	0.069	0.996	0.652

Table SA-49: Fuzzy Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.862	18.862	2022	3994	0.004	0.056	0.056	0.957	0.408
$\mathbf{b}_2$	17.546	17.546	1877	3512	0.004	0.058	0.058	0.953	0.432
$\mathbf{b}_3$	15.909	15.909	1661	2914	0.004	0.063	0.063	0.953	0.471
$\mathbf{b}_4$	13.997	13.997	1382	2257	0.003	0.070	0.070	0.958	0.535
$\mathbf{b}_5$	12.000	12.000	1081	1641	0.001	0.081	0.081	0.961	0.629
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.000	0.097	0.097	0.957	0.774
$\mathbf{b}_7$	8.562	8.562	596	783	0.001	0.118	0.118	0.967	0.956
$\mathbf{b}_8$	8.375	8.375	599	680	0.005	0.124	0.124	0.966	1.027
$\mathbf{b}_9$	8.420	8.420	647	585	0.009	0.124	0.124	0.975	1.048
$\mathbf{b}_{10}$	8.534	8.534	708	483	0.008	0.127	0.127	0.970	1.364
$\mathbf{b}_{11}$	9.351	9.351	877	445	0.004	0.136	0.136	0.977	22.769
$\mathbf{b}_{12}$	8.464	8.464	680	522	-0.005	0.126	0.126	0.970	1.213
$\mathbf{b}_{13}$	8.411	8.411	612	667	-0.003	0.122	0.122	0.970	0.960
$\mathbf{b}_{14}$	8.717	8.717	614	821	0.004	0.110	0.110	0.967	0.867
$\mathbf{b}_{15}$	11.200	11.200	961	1419	0.004	0.083	0.083	0.958	0.631
$\mathbf{b}_{16}$	13.996	13.996	1382	2258	0.002	0.064	0.064	0.954	0.491
$\mathbf{b}_{17}$	16.483	16.483	1739	3123	0.001	0.055	0.055	0.957	0.413
$\mathbf{b}_{18}$	18.112	18.112	1930	3712	0.001	0.050	0.050	0.959	0.379
$\mathbf{b}_{19}$	19.440	19.440	2038	4146	-0.000	0.048	0.048	0.951	0.357
$\mathbf{b}_{20}$	20.668	20.668	2098	4464	-0.002	0.047	0.047	0.954	0.340
$\mathbf{b}_{21}$	21.765	21.765	2107	4658	-0.004	0.046	0.046	0.951	0.329
`Uniform`								0.978	2.660
`WBATE`					0.002	0.033	0.033	0.959	1.263
`LBATE`					0.135	0.087	0.160	0.997	18.630

Table SA-50: Fuzzy Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.558	8.558	609	823	0.005	0.118	0.119	0.956	0.458
$\mathbf{b}_2$	8.391	8.391	598	803	0.005	0.120	0.120	0.952	0.463
$\mathbf{b}_3$	8.281	8.281	589	788	0.004	0.120	0.120	0.958	0.467
$\mathbf{b}_4$	8.239	8.239	583	779	0.003	0.122	0.122	0.959	0.468
$\mathbf{b}_5$	8.251	8.251	579	773	0.001	0.120	0.120	0.959	0.468
$\mathbf{b}_6$	8.325	8.325	577	772	-0.001	0.121	0.121	0.957	0.467
$\mathbf{b}_7$	8.375	8.375	575	752	0.001	0.122	0.122	0.956	0.471
$\mathbf{b}_8$	8.375	8.375	599	680	0.005	0.124	0.124	0.957	0.478
$\mathbf{b}_9$	8.420	8.420	647	585	0.009	0.124	0.124	0.960	0.486
$\mathbf{b}_{10}$	8.534	8.534	708	483	0.008	0.127	0.127	0.962	0.496
$\mathbf{b}_{11}$	9.351	9.351	877	445	0.004	0.136	0.136	0.961	0.519
$\mathbf{b}_{12}$	8.464	8.464	680	522	-0.005	0.126	0.126	0.960	0.486
$\mathbf{b}_{13}$	8.411	8.411	612	667	-0.003	0.122	0.122	0.960	0.462
$\mathbf{b}_{14}$	8.413	8.413	579	768	0.004	0.115	0.115	0.957	0.442
$\mathbf{b}_{15}$	8.321	8.321	583	781	0.005	0.114	0.114	0.956	0.432
$\mathbf{b}_{16}$	8.160	8.160	573	764	0.001	0.113	0.113	0.954	0.433
$\mathbf{b}_{17}$	8.070	8.070	561	747	-0.000	0.110	0.110	0.955	0.432
$\mathbf{b}_{18}$	8.183	8.183	566	756	0.001	0.108	0.108	0.956	0.426
$\mathbf{b}_{19}$	8.539	8.539	590	798	0.002	0.107	0.107	0.951	0.409
$\mathbf{b}_{20}$	9.055	9.055	623	856	-0.000	0.100	0.100	0.952	0.388
$\mathbf{b}_{21}$	9.535	9.535	639	893	-0.002	0.097	0.097	0.951	0.375
`Uniform`								0.972	0.691
`WBATE`					0.002	0.039	0.039	0.953	0.151
`LBATE`					0.179	0.088	0.199	0.997	0.793

Table SA-51: Fuzzy Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.325	19.325	2100	4182	0.002	0.058	0.058	0.961	0.406
$\mathbf{b}_2$	19.171	19.171	2152	4123	0.003	0.059	0.059	0.960	0.406
$\mathbf{b}_3$	19.127	19.127	2235	4046	0.004	0.059	0.060	0.960	0.404
$\mathbf{b}_4$	19.198	19.198	2351	3949	0.006	0.061	0.061	0.959	0.403
$\mathbf{b}_5$	19.404	19.404	2495	3845	0.010	0.061	0.061	0.963	0.400
$\mathbf{b}_6$	19.638	19.638	2635	3703	0.013	0.060	0.061	0.959	0.399
$\mathbf{b}_7$	19.901	19.901	2763	3534	0.015	0.059	0.061	0.957	0.400
$\mathbf{b}_8$	20.219	20.219	2874	3351	0.014	0.060	0.062	0.958	0.400
$\mathbf{b}_9$	20.701	20.701	2983	3189	0.011	0.059	0.060	0.959	0.398
$\mathbf{b}_{10}$	21.556	21.556	3119	3110	0.007	0.058	0.059	0.961	0.397
$\mathbf{b}_{11}$	23.400	23.400	3362	3287	0.002	0.058	0.058	0.961	0.406
$\mathbf{b}_{12}$	21.216	21.216	3072	3143	-0.005	0.059	0.059	0.963	0.392
$\mathbf{b}_{13}$	20.499	20.499	2936	3369	-0.012	0.058	0.060	0.958	0.385
$\mathbf{b}_{14}$	20.118	20.118	2793	3651	-0.013	0.058	0.059	0.955	0.373
$\mathbf{b}_{15}$	19.695	19.695	2591	3862	-0.009	0.058	0.058	0.958	0.369
$\mathbf{b}_{16}$	19.024	19.024	2321	3897	-0.003	0.057	0.057	0.964	0.375
$\mathbf{b}_{17}$	18.414	18.414	2081	3827	0.002	0.055	0.055	0.966	0.380
$\mathbf{b}_{18}$	18.243	18.243	1959	3776	0.003	0.053	0.053	0.961	0.382
$\mathbf{b}_{19}$	18.530	18.530	1924	3797	0.002	0.052	0.052	0.954	0.376
$\mathbf{b}_{20}$	19.104	19.104	1912	3865	0.000	0.052	0.052	0.957	0.369
$\mathbf{b}_{21}$	19.617	19.617	1867	3854	-0.002	0.053	0.053	0.956	0.368
`Uniform`								0.971	0.585
`WBATE`					0.002	0.027	0.027	0.948	0.145
`LBATE`					0.056	0.046	0.072	0.997	0.668

Table SA-52: Fuzzy Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.822	19.822	2159	4374	0.001	0.073	0.073	0.962	0.522
$\mathbf{b}_2$	19.210	19.210	2146	4137	0.001	0.074	0.074	0.962	0.525
$\mathbf{b}_3$	18.737	18.737	2155	3903	0.002	0.074	0.074	0.963	0.526
$\mathbf{b}_4$	18.364	18.364	2185	3658	0.005	0.074	0.074	0.957	0.528
$\mathbf{b}_5$	18.091	18.091	2234	3405	0.009	0.075	0.076	0.957	0.530
$\mathbf{b}_6$	17.917	17.917	2293	3150	0.013	0.076	0.077	0.961	0.531
$\mathbf{b}_7$	17.759	17.759	2342	2876	0.015	0.077	0.078	0.960	0.535
$\mathbf{b}_8$	17.432	17.432	2336	2540	0.015	0.078	0.079	0.958	0.547
$\mathbf{b}_9$	17.290	17.290	2346	2244	0.012	0.078	0.079	0.957	0.553
$\mathbf{b}_{10}$	17.153	17.153	2333	1945	0.007	0.076	0.076	0.959	0.547
$\mathbf{b}_{11}$	18.116	18.116	2499	1885	0.002	0.070	0.070	0.965	0.497
$\mathbf{b}_{12}$	17.198	17.198	2339	2062	-0.005	0.078	0.078	0.962	0.567
$\mathbf{b}_{13}$	17.364	17.364	2332	2471	-0.012	0.082	0.083	0.958	0.570
$\mathbf{b}_{14}$	17.720	17.720	2319	2912	-0.014	0.083	0.084	0.955	0.559
$\mathbf{b}_{15}$	17.870	17.870	2228	3261	-0.010	0.084	0.085	0.954	0.564
$\mathbf{b}_{16}$	18.113	18.113	2137	3576	-0.003	0.083	0.083	0.957	0.572
$\mathbf{b}_{17}$	18.540	18.540	2085	3865	0.001	0.082	0.082	0.955	0.581
$\mathbf{b}_{18}$	19.281	19.281	2097	4170	0.001	0.083	0.083	0.954	0.585
$\mathbf{b}_{19}$	20.431	20.431	2159	4545	-0.000	0.083	0.083	0.954	0.581
$\mathbf{b}_{20}$	21.957	21.957	2239	4992	-0.002	0.082	0.082	0.957	0.574
$\mathbf{b}_{21}$	23.561	23.561	2283	5391	-0.004	0.083	0.083	0.959	0.571
`Uniform`								0.971	0.830
`WBATE`					0.001	0.034	0.034	0.946	0.191
`LBATE`					0.083	0.059	0.102	0.996	0.913

Table SA-53: Fuzzy Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.027	19.027	2044	4060	0.001	0.074	0.074	0.962	0.540
$\mathbf{b}_2$	17.579	17.579	1882	3526	0.002	0.077	0.077	0.959	0.568
$\mathbf{b}_3$	15.902	15.902	1660	2912	0.003	0.081	0.081	0.959	0.613
$\mathbf{b}_4$	13.995	13.995	1382	2257	0.002	0.090	0.090	0.954	0.687
$\mathbf{b}_5$	12.000	12.000	1081	1641	0.000	0.103	0.103	0.957	0.798
$\mathbf{b}_6$	10.000	10.000	788	1112	-0.002	0.123	0.123	0.961	0.966
$\mathbf{b}_7$	8.181	8.181	545	717	-0.003	0.152	0.152	0.966	1.360
$\mathbf{b}_8$	7.642	7.642	495	577	0.002	0.165	0.165	0.974	1.454
$\mathbf{b}_9$	7.575	7.575	521	483	0.009	0.174	0.174	0.971	1.570
$\mathbf{b}_{10}$	7.515	7.515	554	378	0.009	0.171	0.171	0.973	9.809
$\mathbf{b}_{11}$	7.937	7.937	653	311	0.004	0.155	0.155	0.984	1619.334
$\mathbf{b}_{12}$	7.534	7.534	540	421	-0.005	0.173	0.173	0.974	2.352
$\mathbf{b}_{13}$	7.608	7.608	496	557	-0.002	0.173	0.173	0.971	1.508
$\mathbf{b}_{14}$	8.434	8.434	576	769	-0.000	0.155	0.155	0.959	4.695
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.002	0.120	0.120	0.950	0.897
$\mathbf{b}_{16}$	13.991	13.991	1382	2256	-0.001	0.098	0.098	0.952	0.733
$\mathbf{b}_{17}$	16.545	16.545	1748	3147	-0.001	0.087	0.087	0.951	0.644
$\mathbf{b}_{18}$	18.559	18.559	1990	3886	-0.000	0.085	0.085	0.952	0.604
$\mathbf{b}_{19}$	20.288	20.288	2142	4486	-0.000	0.083	0.083	0.953	0.585
$\mathbf{b}_{20}$	21.947	21.947	2238	4988	-0.002	0.082	0.082	0.957	0.574
$\mathbf{b}_{21}$	23.560	23.560	2283	5391	-0.004	0.083	0.083	0.959	0.571
`Uniform`								0.978	119.194
`WBATE`					0.000	0.043	0.043	0.960	77.948
`LBATE`					0.194	0.120	0.228	0.998	1248.167

Table SA-54: Fuzzy Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.684	8.684	625	848	0.001	0.153	0.153	0.959	0.593
$\mathbf{b}_2$	8.416	8.416	601	807	0.002	0.154	0.154	0.958	0.599
$\mathbf{b}_3$	8.209	8.209	580	773	0.003	0.156	0.156	0.958	0.605
$\mathbf{b}_4$	8.045	8.045	559	740	0.001	0.160	0.160	0.955	0.610
$\mathbf{b}_5$	7.926	7.926	539	711	-0.002	0.160	0.160	0.955	0.616
$\mathbf{b}_6$	7.850	7.850	521	684	-0.004	0.161	0.161	0.962	0.619
$\mathbf{b}_7$	7.780	7.780	501	651	-0.003	0.162	0.162	0.958	0.625
$\mathbf{b}_8$	7.637	7.637	495	576	0.002	0.165	0.165	0.961	0.637
$\mathbf{b}_9$	7.575	7.575	521	483	0.009	0.174	0.174	0.954	0.647
$\mathbf{b}_{10}$	7.515	7.515	554	378	0.009	0.171	0.171	0.956	0.643
$\mathbf{b}_{11}$	7.937	7.937	653	311	0.004	0.155	0.155	0.966	0.582
$\mathbf{b}_{12}$	7.534	7.534	540	421	-0.005	0.173	0.173	0.958	0.664
$\mathbf{b}_{13}$	7.607	7.607	496	557	-0.002	0.174	0.174	0.953	0.660
$\mathbf{b}_{14}$	7.763	7.763	500	654	-0.000	0.172	0.172	0.956	0.652
$\mathbf{b}_{15}$	7.829	7.829	524	689	-0.003	0.172	0.172	0.953	0.651
$\mathbf{b}_{16}$	7.935	7.935	546	720	-0.000	0.171	0.171	0.955	0.654
$\mathbf{b}_{17}$	8.122	8.122	568	755	0.000	0.169	0.169	0.954	0.657
$\mathbf{b}_{18}$	8.447	8.447	598	805	-0.001	0.174	0.174	0.946	0.658
$\mathbf{b}_{19}$	8.951	8.951	640	877	-0.001	0.168	0.168	0.952	0.649
$\mathbf{b}_{20}$	9.620	9.620	691	969	0.000	0.162	0.162	0.954	0.637
$\mathbf{b}_{21}$	10.322	10.322	731	1051	-0.000	0.163	0.163	0.956	0.631
`Uniform`								0.970	0.967
`WBATE`					0.001	0.052	0.052	0.947	0.199
`LBATE`					0.261	0.126	0.290	0.996	1.095

Table SA-55: Fuzzy Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.920	18.920	2042	4036	-0.000	0.084	0.084	0.962	0.554
$\mathbf{b}_2$	18.614	18.614	2065	3924	-0.000	0.083	0.083	0.961	0.551
$\mathbf{b}_3$	18.397	18.397	2110	3794	0.000	0.083	0.083	0.963	0.548
$\mathbf{b}_4$	18.284	18.284	2182	3647	0.003	0.083	0.083	0.959	0.543
$\mathbf{b}_5$	18.172	18.172	2256	3454	0.006	0.084	0.084	0.960	0.541
$\mathbf{b}_6$	18.104	18.104	2332	3236	0.010	0.084	0.085	0.966	0.540
$\mathbf{b}_7$	17.715	17.715	2330	2898	0.011	0.086	0.086	0.962	0.551
$\mathbf{b}_8$	17.474	17.474	2340	2587	0.012	0.086	0.087	0.965	0.558
$\mathbf{b}_9$	17.306	17.306	2344	2288	0.010	0.087	0.087	0.959	0.570
$\mathbf{b}_{10}$	17.385	17.385	2370	2040	0.005	0.084	0.084	0.964	0.556
$\mathbf{b}_{11}$	18.282	18.282	2516	1968	0.001	0.076	0.076	0.969	0.509
$\mathbf{b}_{12}$	17.522	17.522	2395	2180	-0.005	0.086	0.086	0.966	0.570
$\mathbf{b}_{13}$	17.903	17.903	2435	2652	-0.011	0.090	0.090	0.961	0.569
$\mathbf{b}_{14}$	18.374	18.374	2449	3140	-0.011	0.091	0.092	0.964	0.556
$\mathbf{b}_{15}$	18.805	18.805	2421	3586	-0.007	0.093	0.093	0.960	0.555
$\mathbf{b}_{16}$	18.919	18.919	2307	3867	-0.002	0.092	0.092	0.962	0.567
$\mathbf{b}_{17}$	18.942	18.942	2175	4021	0.002	0.091	0.092	0.962	0.583
$\mathbf{b}_{18}$	19.136	19.136	2090	4131	0.003	0.096	0.096	0.964	0.600
$\mathbf{b}_{19}$	19.695	19.695	2068	4278	0.003	0.097	0.097	0.958	0.611
$\mathbf{b}_{20}$	20.557	20.557	2072	4464	0.004	0.097	0.097	0.961	0.619
$\mathbf{b}_{21}$	21.470	21.470	2054	4601	0.004	0.100	0.100	0.963	0.635
`Uniform`								0.976	0.854
`WBATE`					0.002	0.037	0.037	0.941	0.195
`LBATE`					0.102	0.077	0.127	0.997	1.032

Table SA-56: Fuzzy Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.945	17.945	1903	3648	0.002	0.060	0.060	0.957	0.443
$\mathbf{b}_2$	17.380	17.380	1868	3459	0.001	0.061	0.061	0.953	0.450
$\mathbf{b}_3$	17.202	17.202	1891	3370	0.002	0.062	0.063	0.960	0.448
$\mathbf{b}_4$	17.380	17.380	1997	3341	0.007	0.063	0.064	0.960	0.439
$\mathbf{b}_5$	17.857	17.857	2186	3339	0.016	0.063	0.065	0.956	0.424
$\mathbf{b}_6$	18.430	18.430	2397	3310	0.026	0.060	0.065	0.950	0.407
$\mathbf{b}_7$	18.889	18.889	2574	3202	0.032	0.057	0.065	0.958	0.394
$\mathbf{b}_8$	18.988	18.988	2653	2961	0.031	0.055	0.063	0.956	0.389
$\mathbf{b}_9$	19.108	19.108	2707	2710	0.025	0.054	0.059	0.957	0.387
$\mathbf{b}_{10}$	19.393	19.393	2761	2487	0.015	0.054	0.056	0.959	0.390
$\mathbf{b}_{11}$	21.303	21.303	3057	2662	0.006	0.053	0.054	0.959	0.391
$\mathbf{b}_{12}$	19.219	19.219	2732	2561	-0.006	0.054	0.054	0.959	0.379
$\mathbf{b}_{13}$	19.004	19.004	2665	2911	-0.018	0.053	0.056	0.953	0.365
$\mathbf{b}_{14}$	18.692	18.692	2519	3196	-0.023	0.052	0.057	0.951	0.358
$\mathbf{b}_{15}$	17.796	17.796	2212	3246	-0.016	0.054	0.056	0.957	0.368
$\mathbf{b}_{16}$	16.806	16.806	1891	3154	-0.005	0.054	0.055	0.958	0.385
$\mathbf{b}_{17}$	16.262	16.262	1722	3052	-0.001	0.053	0.053	0.954	0.395
$\mathbf{b}_{18}$	16.380	16.380	1697	3070	-0.002	0.052	0.052	0.956	0.392
$\mathbf{b}_{19}$	17.047	17.047	1735	3229	-0.002	0.052	0.052	0.952	0.380
$\mathbf{b}_{20}$	17.934	17.934	1774	3415	-0.004	0.051	0.051	0.951	0.369
$\mathbf{b}_{21}$	18.901	18.901	1791	3574	-0.008	0.050	0.051	0.957	0.360
`Uniform`								0.966	0.596
`WBATE`					0.004	0.026	0.026	0.951	0.143
`LBATE`					0.063	0.043	0.076	0.995	0.648

Table SA-57: Fuzzy Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.655	17.655	1862	3533	0.003	0.061	0.061	0.957	0.448
$\mathbf{b}_2$	16.710	16.710	1760	3207	0.002	0.063	0.063	0.952	0.463
$\mathbf{b}_3$	15.625	15.625	1619	2818	0.002	0.065	0.065	0.958	0.485
$\mathbf{b}_4$	13.979	13.979	1379	2252	0.002	0.070	0.070	0.956	0.536
$\mathbf{b}_5$	12.000	12.000	1080	1642	0.002	0.081	0.081	0.958	0.622
$\mathbf{b}_6$	10.000	10.000	787	1114	0.001	0.096	0.096	0.962	0.752
$\mathbf{b}_7$	8.509	8.509	589	775	0.000	0.114	0.114	0.967	0.910
$\mathbf{b}_8$	8.319	8.319	591	673	0.006	0.118	0.118	0.971	0.952
$\mathbf{b}_9$	8.372	8.372	639	580	0.014	0.117	0.118	0.964	0.960
$\mathbf{b}_{10}$	8.496	8.496	702	480	0.013	0.118	0.119	0.971	0.954
$\mathbf{b}_{11}$	9.333	9.333	873	444	0.006	0.123	0.123	0.968	14.087
$\mathbf{b}_{12}$	8.420	8.420	673	518	-0.004	0.115	0.115	0.965	0.905
$\mathbf{b}_{13}$	8.326	8.326	600	655	-0.002	0.110	0.110	0.969	0.871
$\mathbf{b}_{14}$	8.643	8.643	604	807	0.001	0.102	0.102	0.962	0.787
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.076	0.076	0.955	0.581
$\mathbf{b}_{16}$	13.958	13.958	1377	2246	-0.002	0.060	0.060	0.953	0.458
$\mathbf{b}_{17}$	15.667	15.667	1623	2837	-0.002	0.054	0.054	0.954	0.407
$\mathbf{b}_{18}$	16.274	16.274	1681	3028	-0.002	0.053	0.053	0.957	0.394
$\mathbf{b}_{19}$	17.039	17.039	1734	3226	-0.002	0.052	0.052	0.952	0.380
$\mathbf{b}_{20}$	17.934	17.934	1774	3415	-0.004	0.051	0.051	0.951	0.369
$\mathbf{b}_{21}$	18.901	18.901	1791	3574	-0.008	0.050	0.051	0.957	0.360
`Uniform`								0.976	1.936
`WBATE`					0.001	0.032	0.032	0.953	0.819
`LBATE`					0.133	0.079	0.155	0.997	11.582

Table SA-58: Fuzzy Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.862	7.862	525	695	0.002	0.133	0.133	0.952	0.512
$\mathbf{b}_2$	7.614	7.614	504	661	0.001	0.139	0.139	0.952	0.522
$\mathbf{b}_3$	7.536	7.536	498	652	0.001	0.135	0.135	0.960	0.520
$\mathbf{b}_4$	7.614	7.614	507	666	-0.001	0.134	0.134	0.955	0.511
$\mathbf{b}_5$	7.823	7.823	527	697	0.001	0.128	0.127	0.958	0.491
$\mathbf{b}_6$	8.074	8.074	547	729	0.001	0.123	0.123	0.954	0.471
$\mathbf{b}_7$	8.275	8.275	563	736	-0.000	0.120	0.120	0.956	0.458
$\mathbf{b}_8$	8.319	8.319	591	673	0.006	0.118	0.118	0.956	0.454
$\mathbf{b}_9$	8.372	8.372	639	580	0.014	0.117	0.118	0.955	0.451
$\mathbf{b}_{10}$	8.496	8.496	702	480	0.013	0.118	0.119	0.954	0.453
$\mathbf{b}_{11}$	9.333	9.333	873	444	0.006	0.123	0.123	0.956	0.466
$\mathbf{b}_{12}$	8.420	8.420	673	518	-0.004	0.115	0.115	0.951	0.440
$\mathbf{b}_{13}$	8.326	8.326	600	655	-0.002	0.110	0.110	0.955	0.423
$\mathbf{b}_{14}$	8.189	8.189	553	729	0.002	0.110	0.110	0.952	0.416
$\mathbf{b}_{15}$	7.797	7.797	522	687	-0.001	0.110	0.110	0.951	0.428
$\mathbf{b}_{16}$	7.363	7.363	479	620	-0.001	0.117	0.117	0.953	0.447
$\mathbf{b}_{17}$	7.124	7.124	450	579	-0.006	0.118	0.119	0.953	0.459
$\mathbf{b}_{18}$	7.176	7.176	448	577	-0.003	0.115	0.115	0.955	0.454
$\mathbf{b}_{19}$	7.468	7.468	466	607	0.001	0.114	0.114	0.952	0.439
$\mathbf{b}_{20}$	7.857	7.857	486	643	-0.001	0.112	0.112	0.951	0.426
$\mathbf{b}_{21}$	8.281	8.281	501	672	-0.003	0.106	0.106	0.956	0.415
`Uniform`								0.966	0.702
`WBATE`					0.001	0.038	0.038	0.945	0.148
`LBATE`					0.181	0.085	0.200	0.995	0.783

Table SA-59: Fuzzy Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.303	17.303	1816	3409	0.000	0.064	0.064	0.960	0.461
$\mathbf{b}_2$	16.910	16.910	1799	3295	-0.001	0.065	0.065	0.956	0.466
$\mathbf{b}_3$	16.825	16.825	1837	3247	-0.000	0.067	0.067	0.965	0.464
$\mathbf{b}_4$	17.058	17.058	1950	3248	0.004	0.069	0.069	0.965	0.457
$\mathbf{b}_5$	17.578	17.578	2135	3278	0.012	0.070	0.071	0.963	0.441
$\mathbf{b}_6$	18.313	18.313	2368	3312	0.021	0.068	0.071	0.958	0.422
$\mathbf{b}_7$	19.002	19.002	2581	3280	0.028	0.063	0.069	0.957	0.404
$\mathbf{b}_8$	19.652	19.652	2762	3196	0.028	0.059	0.065	0.955	0.388
$\mathbf{b}_9$	20.340	20.340	2915	3094	0.023	0.055	0.060	0.957	0.374
$\mathbf{b}_{10}$	21.331	21.331	3079	3051	0.014	0.054	0.056	0.961	0.364
$\mathbf{b}_{11}$	23.114	23.114	3319	3205	0.005	0.053	0.054	0.960	0.368
$\mathbf{b}_{12}$	20.904	20.904	3020	3053	-0.006	0.053	0.054	0.960	0.357
$\mathbf{b}_{13}$	19.854	19.854	2815	3187	-0.017	0.054	0.057	0.956	0.358
$\mathbf{b}_{14}$	18.695	18.695	2510	3236	-0.020	0.057	0.060	0.952	0.368
$\mathbf{b}_{15}$	17.220	17.220	2101	3104	-0.011	0.061	0.062	0.962	0.390
$\mathbf{b}_{16}$	15.920	15.920	1748	2886	-0.000	0.061	0.061	0.963	0.414
$\mathbf{b}_{17}$	15.326	15.326	1584	2741	0.002	0.058	0.058	0.959	0.422
$\mathbf{b}_{18}$	15.400	15.400	1560	2726	0.001	0.056	0.056	0.958	0.416
$\mathbf{b}_{19}$	15.907	15.907	1584	2816	0.000	0.057	0.057	0.956	0.406
$\mathbf{b}_{20}$	16.537	16.537	1600	2910	-0.002	0.057	0.057	0.959	0.399
$\mathbf{b}_{21}$	17.049	17.049	1573	2921	-0.004	0.058	0.058	0.958	0.399
`Uniform`								0.971	0.613
`WBATE`					0.004	0.027	0.027	0.949	0.145
`LBATE`					0.066	0.043	0.079	0.995	0.678

Table SA-60: Fuzzy Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.213	18.213	1939	3752	-0.000	0.070	0.070	0.957	0.500
$\mathbf{b}_2$	17.557	17.557	1894	3525	0.000	0.070	0.070	0.955	0.499
$\mathbf{b}_3$	17.203	17.203	1893	3370	0.001	0.069	0.069	0.957	0.491
$\mathbf{b}_4$	17.159	17.159	1958	3268	0.006	0.069	0.069	0.962	0.477
$\mathbf{b}_5$	17.317	17.317	2078	3171	0.014	0.068	0.070	0.963	0.459
$\mathbf{b}_6$	17.528	17.528	2213	3041	0.024	0.066	0.070	0.958	0.442
$\mathbf{b}_7$	17.616	17.616	2312	2838	0.030	0.063	0.070	0.954	0.429
$\mathbf{b}_8$	17.424	17.424	2334	2539	0.030	0.062	0.069	0.956	0.426
$\mathbf{b}_9$	17.357	17.357	2359	2261	0.024	0.060	0.065	0.954	0.423
$\mathbf{b}_{10}$	17.274	17.274	2356	1973	0.014	0.058	0.059	0.962	0.413
$\mathbf{b}_{11}$	18.406	18.406	2551	1951	0.004	0.053	0.054	0.959	0.369
$\mathbf{b}_{12}$	17.297	17.297	2359	2086	-0.008	0.060	0.060	0.962	0.426
$\mathbf{b}_{13}$	17.369	17.369	2333	2472	-0.019	0.062	0.065	0.960	0.428
$\mathbf{b}_{14}$	17.534	17.534	2280	2862	-0.021	0.063	0.067	0.959	0.424
$\mathbf{b}_{15}$	17.147	17.147	2080	3047	-0.012	0.067	0.068	0.954	0.441
$\mathbf{b}_{16}$	16.682	16.682	1869	3114	0.000	0.067	0.067	0.962	0.460
$\mathbf{b}_{17}$	16.578	16.578	1771	3166	0.004	0.067	0.067	0.952	0.476
$\mathbf{b}_{18}$	16.978	16.978	1779	3293	0.001	0.066	0.066	0.953	0.476
$\mathbf{b}_{19}$	17.884	17.884	1843	3543	-0.002	0.066	0.066	0.951	0.468
$\mathbf{b}_{20}$	18.952	18.952	1897	3799	-0.005	0.066	0.066	0.946	0.462
$\mathbf{b}_{21}$	20.130	20.130	1928	4034	-0.008	0.066	0.066	0.950	0.457
`Uniform`								0.969	0.680
`WBATE`					0.004	0.028	0.028	0.949	0.153
`LBATE`					0.074	0.047	0.087	0.995	0.729

Table SA-61: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.887	17.887	1892	3622	0.000	0.071	0.071	0.958	0.506
$\mathbf{b}_2$	16.808	16.808	1774	3244	0.001	0.071	0.071	0.953	0.516
$\mathbf{b}_3$	15.591	15.591	1615	2807	0.002	0.073	0.073	0.954	0.534
$\mathbf{b}_4$	13.963	13.963	1377	2248	0.002	0.077	0.077	0.956	0.576
$\mathbf{b}_5$	12.000	12.000	1082	1642	0.002	0.086	0.086	0.961	0.653
$\mathbf{b}_6$	10.000	10.000	788	1113	0.002	0.099	0.099	0.958	0.776
$\mathbf{b}_7$	8.168	8.168	543	715	0.001	0.122	0.122	0.964	0.970
$\mathbf{b}_8$	7.637	7.637	494	577	0.004	0.131	0.131	0.968	1.065
$\mathbf{b}_9$	7.604	7.604	526	487	0.011	0.130	0.130	0.970	1.065
$\mathbf{b}_{10}$	7.568	7.568	562	384	0.010	0.125	0.126	0.970	1.300
$\mathbf{b}_{11}$	8.064	8.064	672	322	0.003	0.112	0.112	0.982	11.299
$\mathbf{b}_{12}$	7.578	7.578	547	425	-0.006	0.129	0.129	0.966	2.008
$\mathbf{b}_{13}$	7.610	7.610	496	558	-0.005	0.128	0.128	0.967	1.014
$\mathbf{b}_{14}$	8.430	8.430	576	769	-0.001	0.115	0.115	0.958	0.896
$\mathbf{b}_{15}$	11.200	11.200	962	1418	0.000	0.089	0.089	0.950	0.666
$\mathbf{b}_{16}$	13.926	13.926	1372	2236	0.002	0.074	0.074	0.955	0.542
$\mathbf{b}_{17}$	15.794	15.794	1641	2883	0.003	0.069	0.069	0.951	0.494
$\mathbf{b}_{18}$	16.820	16.820	1756	3230	0.001	0.067	0.067	0.954	0.480
$\mathbf{b}_{19}$	17.866	17.866	1841	3536	-0.002	0.066	0.066	0.951	0.468
$\mathbf{b}_{20}$	18.951	18.951	1897	3799	-0.005	0.066	0.066	0.946	0.462
$\mathbf{b}_{21}$	20.130	20.130	1928	4034	-0.008	0.066	0.066	0.950	0.457
`Uniform`								0.975	1.943
`WBATE`					0.001	0.034	0.034	0.961	0.759
`LBATE`					0.149	0.086	0.173	0.997	10.563

Table SA-62: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.979	7.979	540	716	-0.001	0.149	0.149	0.955	0.573
$\mathbf{b}_2$	7.692	7.692	514	675	0.002	0.149	0.149	0.954	0.573
$\mathbf{b}_3$	7.537	7.537	499	652	0.002	0.144	0.144	0.957	0.566
$\mathbf{b}_4$	7.517	7.517	497	648	0.000	0.144	0.144	0.953	0.549
$\mathbf{b}_5$	7.587	7.587	500	653	0.000	0.139	0.139	0.955	0.529
$\mathbf{b}_6$	7.679	7.679	501	656	0.001	0.132	0.132	0.956	0.512
$\mathbf{b}_7$	7.718	7.718	493	643	0.000	0.132	0.132	0.954	0.502
$\mathbf{b}_8$	7.634	7.634	494	576	0.004	0.131	0.131	0.958	0.498
$\mathbf{b}_9$	7.604	7.604	526	487	0.011	0.130	0.130	0.955	0.494
$\mathbf{b}_{10}$	7.568	7.568	562	384	0.010	0.125	0.126	0.959	0.481
$\mathbf{b}_{11}$	8.064	8.064	672	322	0.003	0.112	0.112	0.963	0.425
$\mathbf{b}_{12}$	7.578	7.578	547	425	-0.006	0.129	0.129	0.955	0.497
$\mathbf{b}_{13}$	7.609	7.609	496	558	-0.005	0.127	0.128	0.959	0.495
$\mathbf{b}_{14}$	7.682	7.682	490	642	-0.000	0.128	0.128	0.955	0.494
$\mathbf{b}_{15}$	7.512	7.512	488	636	-0.000	0.135	0.135	0.949	0.507
$\mathbf{b}_{16}$	7.309	7.309	472	612	0.005	0.135	0.135	0.954	0.524
$\mathbf{b}_{17}$	7.263	7.263	466	603	0.007	0.142	0.142	0.948	0.540
$\mathbf{b}_{18}$	7.438	7.438	478	622	0.001	0.139	0.139	0.949	0.539
$\mathbf{b}_{19}$	7.835	7.835	508	670	-0.003	0.135	0.135	0.953	0.527
$\mathbf{b}_{20}$	8.303	8.303	537	720	-0.000	0.135	0.135	0.947	0.519
$\mathbf{b}_{21}$	8.819	8.819	560	766	-0.000	0.132	0.132	0.951	0.512
`Uniform`								0.964	0.791
`WBATE`					0.001	0.041	0.041	0.954	0.160
`LBATE`					0.205	0.092	0.225	0.995	0.873

Table SA-63: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.349	17.349	1823	3432	-0.002	0.078	0.078	0.961	0.527
$\mathbf{b}_2$	16.879	16.879	1796	3288	-0.002	0.076	0.076	0.959	0.524
$\mathbf{b}_3$	16.600	16.600	1802	3173	-0.002	0.076	0.076	0.964	0.516
$\mathbf{b}_4$	16.606	16.606	1872	3104	0.002	0.078	0.078	0.963	0.505
$\mathbf{b}_5$	16.893	16.893	2007	3068	0.009	0.078	0.078	0.966	0.487
$\mathbf{b}_6$	17.267	17.267	2163	3001	0.018	0.076	0.078	0.959	0.465
$\mathbf{b}_7$	17.265	17.265	2238	2782	0.023	0.074	0.078	0.955	0.453
$\mathbf{b}_8$	17.338	17.338	2311	2557	0.024	0.071	0.075	0.956	0.442
$\mathbf{b}_9$	17.379	17.379	2356	2307	0.020	0.067	0.070	0.960	0.435
$\mathbf{b}_{10}$	17.542	17.542	2399	2077	0.012	0.063	0.064	0.966	0.418
$\mathbf{b}_{11}$	18.603	18.603	2573	2042	0.003	0.057	0.057	0.964	0.377
$\mathbf{b}_{12}$	17.562	17.562	2403	2187	-0.008	0.065	0.065	0.965	0.431
$\mathbf{b}_{13}$	17.605	17.605	2376	2569	-0.016	0.067	0.069	0.962	0.431
$\mathbf{b}_{14}$	17.447	17.447	2262	2877	-0.015	0.072	0.073	0.961	0.438
$\mathbf{b}_{15}$	16.875	16.875	2039	3004	-0.005	0.077	0.077	0.962	0.461
$\mathbf{b}_{16}$	16.081	16.081	1785	2944	0.006	0.076	0.076	0.968	0.488
$\mathbf{b}_{17}$	15.755	15.755	1653	2902	0.009	0.075	0.075	0.953	0.507
$\mathbf{b}_{18}$	15.961	15.961	1637	2941	0.006	0.073	0.074	0.956	0.508
$\mathbf{b}_{19}$	16.555	16.555	1669	3063	0.003	0.074	0.074	0.954	0.504
$\mathbf{b}_{20}$	17.249	17.249	1687	3182	0.001	0.076	0.076	0.949	0.506
$\mathbf{b}_{21}$	17.956	17.956	1676	3259	0.000	0.077	0.077	0.957	0.512
`Uniform`								0.977	0.717
`WBATE`					0.004	0.030	0.030	0.949	0.158
`LBATE`					0.083	0.055	0.100	0.997	0.814

Table SA-64: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth, ITT Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.940	18.940	2038	4030	0.000	0.034	0.034	0.947	0.248
$\mathbf{b}_2$	18.560	18.560	2046	3894	0.000	0.035	0.035	0.953	0.251
$\mathbf{b}_3$	18.341	18.341	2087	3764	0.000	0.035	0.035	0.948	0.253
$\mathbf{b}_4$	18.247	18.247	2164	3620	0.002	0.036	0.036	0.951	0.254
$\mathbf{b}_5$	18.265	18.265	2269	3460	0.004	0.036	0.036	0.955	0.255
$\mathbf{b}_6$	18.404	18.404	2394	3295	0.006	0.036	0.036	0.956	0.254
$\mathbf{b}_7$	18.492	18.492	2494	3084	0.007	0.036	0.037	0.954	0.256
$\mathbf{b}_8$	18.409	18.409	2536	2801	0.007	0.037	0.038	0.952	0.262
$\mathbf{b}_9$	18.445	18.445	2577	2534	0.006	0.038	0.038	0.951	0.269
$\mathbf{b}_{10}$	18.570	18.570	2606	2280	0.004	0.039	0.040	0.952	0.280
$\mathbf{b}_{11}$	20.183	20.183	2867	2376	0.000	0.040	0.040	0.954	0.297
$\mathbf{b}_{12}$	18.471	18.471	2589	2371	-0.005	0.039	0.039	0.952	0.275
$\mathbf{b}_{13}$	18.461	18.461	2555	2759	-0.011	0.037	0.039	0.950	0.263
$\mathbf{b}_{14}$	18.567	18.567	2494	3154	-0.012	0.036	0.038	0.947	0.255
$\mathbf{b}_{15}$	18.485	18.485	2354	3449	-0.008	0.036	0.037	0.948	0.251
$\mathbf{b}_{16}$	18.323	18.323	2178	3645	-0.003	0.036	0.036	0.954	0.252
$\mathbf{b}_{17}$	18.304	18.304	2048	3779	0.000	0.035	0.035	0.956	0.253
$\mathbf{b}_{18}$	18.620	18.620	2006	3912	0.001	0.034	0.034	0.956	0.252
$\mathbf{b}_{19}$	19.300	19.300	2021	4091	0.001	0.033	0.033	0.949	0.247
$\mathbf{b}_{20}$	20.308	20.308	2057	4323	0.000	0.033	0.033	0.955	0.241
$\mathbf{b}_{21}$	21.402	21.402	2067	4519	-0.000	0.033	0.033	0.950	0.236
`Uniform`								0.949	0.386
`WBATE`					-0.000	0.017	0.017	0.948	0.095
`LBATE`					0.044	0.026	0.052	0.981	0.399

Table SA-65: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.446	18.446	1967	3834	0.001	0.034	0.034	0.947	0.253
$\mathbf{b}_2$	17.301	17.301	1843	3422	0.001	0.036	0.036	0.948	0.266
$\mathbf{b}_3$	15.833	15.833	1650	2887	0.001	0.039	0.039	0.949	0.288
$\mathbf{b}_4$	13.993	13.993	1382	2255	0.001	0.044	0.044	0.950	0.324
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.000	0.050	0.050	0.943	0.378
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.002	0.059	0.059	0.943	0.454
$\mathbf{b}_7$	8.375	8.375	570	751	-0.002	0.072	0.072	0.945	0.546
$\mathbf{b}_8$	8.066	8.066	554	636	0.000	0.079	0.079	0.936	0.580
$\mathbf{b}_9$	8.081	8.081	595	543	0.003	0.080	0.080	0.942	0.597
$\mathbf{b}_{10}$	8.136	8.136	646	441	0.003	0.083	0.083	0.940	0.614
$\mathbf{b}_{11}$	8.842	8.842	794	394	-0.001	0.091	0.091	0.930	0.677
$\mathbf{b}_{12}$	8.092	8.092	623	480	-0.006	0.081	0.081	0.943	0.608
$\mathbf{b}_{13}$	8.088	8.088	564	622	-0.004	0.079	0.079	0.946	0.582
$\mathbf{b}_{14}$	8.563	8.563	593	793	0.001	0.072	0.072	0.939	0.533
$\mathbf{b}_{15}$	11.200	11.200	961	1419	0.002	0.054	0.054	0.943	0.403
$\mathbf{b}_{16}$	13.993	13.993	1381	2257	0.001	0.043	0.043	0.945	0.323
$\mathbf{b}_{17}$	16.467	16.467	1737	3117	0.000	0.037	0.037	0.949	0.277
$\mathbf{b}_{18}$	18.090	18.090	1927	3704	0.000	0.034	0.034	0.955	0.257
$\mathbf{b}_{19}$	19.259	19.259	2016	4074	0.001	0.033	0.033	0.949	0.247
$\mathbf{b}_{20}$	20.308	20.308	2057	4323	0.000	0.033	0.033	0.955	0.241
$\mathbf{b}_{21}$	21.402	21.402	2067	4519	-0.000	0.033	0.033	0.950	0.236
`Uniform`								0.925	0.633
`WBATE`					-0.000	0.021	0.021	0.950	0.122
`LBATE`					0.094	0.049	0.106	0.966	0.783

Table SA-66: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.298	8.298	578	774	0.001	0.075	0.075	0.942	0.286
$\mathbf{b}_2$	8.131	8.131	566	754	0.001	0.076	0.076	0.942	0.290
$\mathbf{b}_3$	8.035	8.035	559	741	0.001	0.076	0.076	0.951	0.293
$\mathbf{b}_4$	7.994	7.994	553	733	0.000	0.076	0.076	0.946	0.294
$\mathbf{b}_5$	8.002	8.002	549	726	-0.001	0.075	0.075	0.947	0.295
$\mathbf{b}_6$	8.063	8.063	546	723	-0.003	0.075	0.075	0.949	0.294
$\mathbf{b}_7$	8.101	8.101	540	706	-0.002	0.075	0.075	0.951	0.296
$\mathbf{b}_8$	8.065	8.065	554	636	0.000	0.079	0.079	0.947	0.303
$\mathbf{b}_9$	8.081	8.081	595	543	0.003	0.080	0.080	0.949	0.311
$\mathbf{b}_{10}$	8.136	8.136	646	441	0.003	0.083	0.083	0.950	0.324
$\mathbf{b}_{11}$	8.842	8.842	794	394	-0.001	0.091	0.091	0.944	0.351
$\mathbf{b}_{12}$	8.092	8.092	623	480	-0.006	0.081	0.081	0.951	0.318
$\mathbf{b}_{13}$	8.088	8.088	564	622	-0.004	0.079	0.079	0.950	0.304
$\mathbf{b}_{14}$	8.134	8.134	544	719	0.001	0.077	0.077	0.946	0.294
$\mathbf{b}_{15}$	8.098	8.098	556	739	0.002	0.076	0.076	0.949	0.290
$\mathbf{b}_{16}$	8.028	8.028	556	738	0.000	0.075	0.075	0.950	0.292
$\mathbf{b}_{17}$	8.019	8.019	555	737	-0.001	0.075	0.075	0.948	0.292
$\mathbf{b}_{18}$	8.157	8.157	562	751	-0.001	0.074	0.074	0.951	0.290
$\mathbf{b}_{19}$	8.456	8.456	580	782	-0.001	0.074	0.074	0.945	0.284
$\mathbf{b}_{20}$	8.897	8.897	604	827	-0.001	0.071	0.071	0.953	0.277
$\mathbf{b}_{21}$	9.376	9.376	621	865	-0.001	0.070	0.070	0.948	0.271
`Uniform`								0.942	0.454
`WBATE`					-0.000	0.025	0.025	0.951	0.097
`LBATE`					0.127	0.048	0.136	0.979	0.468

Table SA-67: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.446	18.446	1976	3846	-0.001	0.036	0.036	0.951	0.257
$\mathbf{b}_2$	18.337	18.337	2020	3819	-0.001	0.037	0.037	0.951	0.257
$\mathbf{b}_3$	18.348	18.348	2101	3775	-0.000	0.037	0.037	0.951	0.257
$\mathbf{b}_4$	18.488	18.488	2219	3716	0.001	0.038	0.038	0.952	0.256
$\mathbf{b}_5$	18.737	18.737	2365	3637	0.002	0.038	0.038	0.954	0.255
$\mathbf{b}_6$	18.966	18.966	2501	3501	0.004	0.038	0.038	0.955	0.253
$\mathbf{b}_7$	19.098	19.098	2604	3300	0.006	0.038	0.039	0.953	0.255
$\mathbf{b}_8$	19.288	19.288	2695	3089	0.006	0.039	0.040	0.953	0.257
$\mathbf{b}_9$	19.613	19.613	2782	2892	0.005	0.039	0.039	0.953	0.260
$\mathbf{b}_{10}$	20.205	20.205	2884	2747	0.003	0.039	0.039	0.952	0.265
$\mathbf{b}_{11}$	21.806	21.806	3110	2852	-0.000	0.040	0.040	0.954	0.281
$\mathbf{b}_{12}$	19.941	19.941	2845	2797	-0.006	0.039	0.040	0.955	0.262
$\mathbf{b}_{13}$	19.520	19.520	2749	3093	-0.010	0.039	0.040	0.953	0.255
$\mathbf{b}_{14}$	19.298	19.298	2631	3407	-0.011	0.039	0.040	0.947	0.251
$\mathbf{b}_{15}$	19.009	19.009	2456	3648	-0.007	0.039	0.040	0.954	0.252
$\mathbf{b}_{16}$	18.515	18.515	2227	3730	-0.002	0.039	0.039	0.955	0.257
$\mathbf{b}_{17}$	18.115	18.115	2033	3722	0.002	0.038	0.038	0.960	0.261
$\mathbf{b}_{18}$	18.047	18.047	1931	3705	0.002	0.036	0.036	0.957	0.262
$\mathbf{b}_{19}$	18.280	18.280	1892	3705	0.002	0.036	0.036	0.948	0.261
$\mathbf{b}_{20}$	18.825	18.825	1878	3762	0.002	0.037	0.037	0.954	0.260
$\mathbf{b}_{21}$	19.374	19.374	1839	3766	0.001	0.038	0.038	0.952	0.261
`Uniform`								0.947	0.389
`WBATE`					-0.000	0.017	0.017	0.948	0.095
`LBATE`					0.050	0.031	0.058	0.978	0.424

Table SA-68: Intention-to-Treat Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.488	19.488	2114	4244	-0.001	0.042	0.042	0.952	0.305
$\mathbf{b}_2$	18.846	18.846	2089	4002	-0.001	0.043	0.043	0.957	0.309
$\mathbf{b}_3$	18.345	18.345	2087	3766	-0.000	0.043	0.043	0.958	0.312
$\mathbf{b}_4$	17.972	17.972	2110	3530	0.001	0.044	0.044	0.957	0.314
$\mathbf{b}_5$	17.684	17.684	2151	3279	0.003	0.045	0.045	0.953	0.316
$\mathbf{b}_6$	17.475	17.475	2202	3021	0.005	0.045	0.045	0.953	0.318
$\mathbf{b}_7$	17.293	17.293	2244	2747	0.007	0.046	0.046	0.951	0.321
$\mathbf{b}_8$	16.916	16.916	2229	2406	0.007	0.047	0.048	0.956	0.330
$\mathbf{b}_9$	16.732	16.732	2233	2109	0.006	0.048	0.048	0.950	0.337
$\mathbf{b}_{10}$	16.448	16.448	2194	1787	0.003	0.049	0.049	0.950	0.343
$\mathbf{b}_{11}$	16.857	16.857	2265	1614	0.000	0.049	0.049	0.948	0.345
$\mathbf{b}_{12}$	16.613	16.613	2223	1927	-0.005	0.049	0.049	0.952	0.351
$\mathbf{b}_{13}$	16.893	16.893	2234	2351	-0.010	0.050	0.051	0.953	0.351
$\mathbf{b}_{14}$	17.334	17.334	2239	2803	-0.011	0.051	0.052	0.952	0.347
$\mathbf{b}_{15}$	17.565	17.565	2165	3168	-0.008	0.052	0.053	0.947	0.352
$\mathbf{b}_{16}$	17.929	17.929	2101	3515	-0.003	0.052	0.052	0.951	0.357
$\mathbf{b}_{17}$	18.505	18.505	2078	3852	0.000	0.051	0.051	0.956	0.362
$\mathbf{b}_{18}$	19.329	19.329	2103	4188	0.001	0.052	0.052	0.951	0.366
$\mathbf{b}_{19}$	20.492	20.492	2167	4569	0.001	0.052	0.052	0.953	0.366
$\mathbf{b}_{20}$	21.968	21.968	2239	4998	0.001	0.053	0.053	0.953	0.366
$\mathbf{b}_{21}$	23.598	23.598	2285	5410	0.000	0.054	0.054	0.955	0.368
`Uniform`								0.946	0.512
`WBATE`					-0.000	0.021	0.021	0.948	0.118
`LBATE`					0.065	0.037	0.075	0.979	0.542

Table SA-69: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.811	18.811	2015	3976	-0.000	0.043	0.043	0.951	0.314
$\mathbf{b}_2$	17.441	17.441	1863	3475	0.000	0.045	0.045	0.954	0.330
$\mathbf{b}_3$	15.849	15.849	1652	2894	0.001	0.047	0.047	0.954	0.355
$\mathbf{b}_4$	13.990	13.990	1381	2255	0.001	0.053	0.053	0.948	0.396
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.001	0.061	0.061	0.946	0.456
$\mathbf{b}_6$	10.000	10.000	788	1112	-0.002	0.072	0.072	0.945	0.542
$\mathbf{b}_7$	8.099	8.099	534	703	-0.003	0.090	0.090	0.944	0.666
$\mathbf{b}_8$	7.421	7.421	465	547	0.000	0.099	0.099	0.946	0.734
$\mathbf{b}_9$	7.330	7.330	487	456	0.002	0.104	0.104	0.931	0.748
$\mathbf{b}_{10}$	7.206	7.206	510	349	0.002	0.104	0.104	0.937	0.772
$\mathbf{b}_{11}$	7.385	7.385	572	266	-0.001	0.106	0.106	0.932	0.769
$\mathbf{b}_{12}$	7.278	7.278	504	395	-0.006	0.104	0.104	0.943	0.780
$\mathbf{b}_{13}$	7.402	7.402	469	531	-0.002	0.105	0.105	0.940	0.775
$\mathbf{b}_{14}$	8.414	8.414	574	766	0.000	0.094	0.094	0.938	0.690
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.074	0.074	0.939	0.535
$\mathbf{b}_{16}$	13.989	13.989	1381	2256	-0.000	0.061	0.061	0.941	0.448
$\mathbf{b}_{17}$	16.557	16.557	1750	3151	-0.000	0.054	0.054	0.949	0.399
$\mathbf{b}_{18}$	18.615	18.615	1998	3908	0.001	0.053	0.053	0.949	0.377
$\mathbf{b}_{19}$	20.332	20.332	2147	4503	0.001	0.053	0.053	0.952	0.369
$\mathbf{b}_{20}$	21.947	21.947	2236	4989	0.001	0.053	0.053	0.953	0.367
$\mathbf{b}_{21}$	23.593	23.593	2284	5408	0.000	0.054	0.054	0.955	0.368
`Uniform`								0.923	0.816
`WBATE`					-0.000	0.026	0.026	0.946	0.149
`LBATE`					0.127	0.061	0.140	0.966	1.001

Table SA-70: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.538	8.538	607	820	-0.001	0.089	0.089	0.949	0.347
$\mathbf{b}_2$	8.257	8.257	582	778	0.000	0.090	0.090	0.950	0.353
$\mathbf{b}_3$	8.037	8.037	559	741	0.001	0.092	0.092	0.952	0.357
$\mathbf{b}_4$	7.874	7.874	538	709	-0.000	0.094	0.094	0.951	0.361
$\mathbf{b}_5$	7.747	7.747	518	679	-0.002	0.095	0.095	0.952	0.365
$\mathbf{b}_6$	7.656	7.656	498	650	-0.004	0.096	0.096	0.951	0.368
$\mathbf{b}_7$	7.576	7.576	477	618	-0.003	0.096	0.096	0.950	0.374
$\mathbf{b}_8$	7.411	7.411	465	546	0.000	0.099	0.099	0.952	0.384
$\mathbf{b}_9$	7.330	7.330	487	456	0.002	0.104	0.104	0.947	0.391
$\mathbf{b}_{10}$	7.206	7.206	510	349	0.002	0.104	0.104	0.942	0.399
$\mathbf{b}_{11}$	7.385	7.385	572	266	-0.001	0.106	0.106	0.945	0.401
$\mathbf{b}_{12}$	7.278	7.278	504	395	-0.006	0.104	0.104	0.952	0.408
$\mathbf{b}_{13}$	7.401	7.401	468	530	-0.002	0.105	0.105	0.949	0.406
$\mathbf{b}_{14}$	7.594	7.594	480	625	-0.000	0.105	0.105	0.947	0.402
$\mathbf{b}_{15}$	7.695	7.695	508	665	-0.002	0.106	0.106	0.945	0.403
$\mathbf{b}_{16}$	7.855	7.855	536	705	0.000	0.105	0.105	0.944	0.407
$\mathbf{b}_{17}$	8.107	8.107	566	752	0.000	0.104	0.104	0.952	0.410
$\mathbf{b}_{18}$	8.468	8.468	601	809	-0.001	0.107	0.107	0.946	0.411
$\mathbf{b}_{19}$	8.978	8.978	644	883	-0.001	0.106	0.106	0.946	0.410
$\mathbf{b}_{20}$	9.624	9.624	692	970	0.001	0.105	0.105	0.946	0.408
$\mathbf{b}_{21}$	10.338	10.338	734	1056	0.001	0.106	0.106	0.948	0.408
`Uniform`								0.938	0.595
`WBATE`					-0.001	0.031	0.031	0.946	0.121
`LBATE`					0.172	0.067	0.185	0.972	0.623

Table SA-71: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.343	18.343	1960	3818	-0.002	0.049	0.049	0.957	0.327
$\mathbf{b}_2$	18.051	18.051	1977	3719	-0.002	0.049	0.049	0.959	0.326
$\mathbf{b}_3$	17.855	17.855	2018	3608	-0.002	0.049	0.049	0.957	0.325
$\mathbf{b}_4$	17.785	17.785	2090	3485	-0.001	0.049	0.049	0.956	0.323
$\mathbf{b}_5$	17.710	17.710	2166	3313	0.001	0.050	0.050	0.956	0.323
$\mathbf{b}_6$	17.660	17.660	2243	3107	0.003	0.050	0.050	0.957	0.322
$\mathbf{b}_7$	17.157	17.157	2217	2745	0.004	0.052	0.052	0.953	0.331
$\mathbf{b}_8$	16.803	16.803	2205	2415	0.005	0.052	0.053	0.959	0.340
$\mathbf{b}_9$	16.462	16.462	2176	2085	0.005	0.054	0.055	0.952	0.351
$\mathbf{b}_{10}$	16.219	16.219	2146	1778	0.002	0.055	0.055	0.951	0.357
$\mathbf{b}_{11}$	16.492	16.492	2191	1581	-0.000	0.054	0.054	0.950	0.361
$\mathbf{b}_{12}$	16.520	16.520	2200	1947	-0.005	0.054	0.054	0.954	0.361
$\mathbf{b}_{13}$	17.243	17.243	2302	2481	-0.009	0.055	0.056	0.956	0.353
$\mathbf{b}_{14}$	17.877	17.877	2349	2999	-0.009	0.056	0.057	0.955	0.346
$\mathbf{b}_{15}$	18.466	18.466	2354	3483	-0.006	0.058	0.058	0.951	0.346
$\mathbf{b}_{16}$	18.673	18.673	2263	3788	-0.002	0.058	0.058	0.955	0.355
$\mathbf{b}_{17}$	18.855	18.855	2164	3992	0.002	0.058	0.058	0.961	0.365
$\mathbf{b}_{18}$	19.160	19.160	2095	4145	0.003	0.061	0.061	0.959	0.377
$\mathbf{b}_{19}$	19.705	19.705	2067	4290	0.003	0.062	0.062	0.954	0.387
$\mathbf{b}_{20}$	20.543	20.543	2067	4467	0.004	0.062	0.063	0.954	0.397
$\mathbf{b}_{21}$	21.462	21.462	2050	4606	0.004	0.065	0.065	0.960	0.409
`Uniform`								0.947	0.531
`WBATE`					-0.000	0.023	0.023	0.941	0.120
`LBATE`					0.080	0.050	0.095	0.976	0.617

Table SA-72: Intention-to-Treat Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.240	17.240	1807	3381	-0.002	0.037	0.037	0.947	0.272
$\mathbf{b}_2$	16.756	16.756	1775	3231	-0.002	0.038	0.038	0.948	0.277
$\mathbf{b}_3$	16.586	16.586	1790	3156	-0.001	0.038	0.038	0.956	0.279
$\mathbf{b}_4$	16.714	16.714	1873	3125	0.001	0.039	0.039	0.954	0.277
$\mathbf{b}_5$	17.111	17.111	2036	3110	0.007	0.040	0.040	0.952	0.271
$\mathbf{b}_6$	17.598	17.598	2226	3064	0.014	0.039	0.042	0.952	0.266
$\mathbf{b}_7$	18.024	18.024	2395	2954	0.020	0.038	0.043	0.946	0.263
$\mathbf{b}_8$	18.133	18.133	2479	2728	0.021	0.037	0.042	0.946	0.266
$\mathbf{b}_9$	18.233	18.233	2535	2481	0.017	0.037	0.041	0.946	0.271
$\mathbf{b}_{10}$	18.423	18.423	2578	2245	0.009	0.039	0.040	0.951	0.281
$\mathbf{b}_{11}$	20.110	20.110	2855	2358	-0.001	0.039	0.039	0.953	0.296
$\mathbf{b}_{12}$	18.393	18.393	2574	2351	-0.008	0.038	0.039	0.952	0.275
$\mathbf{b}_{13}$	18.366	18.366	2537	2735	-0.015	0.037	0.040	0.949	0.262
$\mathbf{b}_{14}$	18.343	18.343	2448	3092	-0.017	0.037	0.041	0.947	0.256
$\mathbf{b}_{15}$	17.668	17.668	2187	3205	-0.012	0.038	0.040	0.952	0.262
$\mathbf{b}_{16}$	16.748	16.748	1880	3135	-0.004	0.039	0.039	0.951	0.274
$\mathbf{b}_{17}$	16.162	16.162	1707	3017	-0.002	0.038	0.038	0.952	0.284
$\mathbf{b}_{18}$	16.176	16.176	1668	2995	-0.001	0.038	0.038	0.950	0.285
$\mathbf{b}_{19}$	16.736	16.736	1694	3114	-0.001	0.038	0.038	0.941	0.281
$\mathbf{b}_{20}$	17.551	17.551	1727	3273	-0.003	0.038	0.038	0.941	0.274
$\mathbf{b}_{21}$	18.491	18.491	1744	3422	-0.005	0.037	0.038	0.948	0.269
`Uniform`								0.946	0.412
`WBATE`					0.001	0.018	0.018	0.951	0.097
`LBATE`					0.032	0.029	0.043	0.990	0.424

Table SA-73: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.072	17.072	1783	3314	-0.001	0.037	0.037	0.947	0.274
$\mathbf{b}_2$	16.314	16.314	1705	3064	-0.001	0.038	0.038	0.948	0.282
$\mathbf{b}_3$	15.415	15.415	1589	2746	-0.001	0.040	0.040	0.954	0.296
$\mathbf{b}_4$	13.947	13.947	1374	2243	-0.000	0.043	0.043	0.948	0.325
$\mathbf{b}_5$	12.000	12.000	1080	1642	0.000	0.050	0.050	0.950	0.377
$\mathbf{b}_6$	10.000	10.000	787	1114	0.000	0.060	0.060	0.950	0.453
$\mathbf{b}_7$	8.292	8.292	559	737	-0.000	0.075	0.075	0.939	0.551
$\mathbf{b}_8$	7.946	7.946	537	620	0.003	0.080	0.080	0.942	0.590
$\mathbf{b}_9$	7.988	7.988	582	533	0.009	0.081	0.081	0.939	0.599
$\mathbf{b}_{10}$	8.071	8.071	636	435	0.008	0.082	0.083	0.941	0.617
$\mathbf{b}_{11}$	8.811	8.811	788	391	0.001	0.091	0.091	0.929	0.676
$\mathbf{b}_{12}$	8.058	8.058	618	477	-0.004	0.081	0.082	0.943	0.606
$\mathbf{b}_{13}$	8.047	8.047	558	616	-0.001	0.078	0.078	0.946	0.580
$\mathbf{b}_{14}$	8.537	8.537	590	788	0.001	0.071	0.071	0.945	0.531
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.054	0.054	0.945	0.401
$\mathbf{b}_{16}$	13.954	13.954	1376	2245	-0.002	0.043	0.043	0.946	0.322
$\mathbf{b}_{17}$	15.604	15.604	1614	2816	-0.002	0.039	0.039	0.951	0.291
$\mathbf{b}_{18}$	16.095	16.095	1656	2963	-0.002	0.038	0.038	0.951	0.286
$\mathbf{b}_{19}$	16.730	16.730	1693	3111	-0.001	0.038	0.038	0.941	0.281
$\mathbf{b}_{20}$	17.551	17.551	1727	3273	-0.003	0.038	0.038	0.941	0.274
$\mathbf{b}_{21}$	18.491	18.491	1744	3422	-0.005	0.037	0.038	0.948	0.269
`Uniform`								0.929	0.648
`WBATE`					-0.000	0.022	0.022	0.947	0.123
`LBATE`					0.090	0.053	0.105	0.978	0.823

Table SA-74: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.553	7.553	489	640	0.001	0.082	0.082	0.946	0.315
$\mathbf{b}_2$	7.341	7.341	472	613	0.001	0.085	0.085	0.942	0.321
$\mathbf{b}_3$	7.267	7.267	467	605	-0.000	0.083	0.083	0.948	0.323
$\mathbf{b}_4$	7.322	7.322	473	615	-0.001	0.083	0.083	0.948	0.321
$\mathbf{b}_5$	7.496	7.496	489	639	-0.000	0.081	0.081	0.950	0.315
$\mathbf{b}_6$	7.710	7.710	505	664	-0.000	0.079	0.079	0.951	0.308
$\mathbf{b}_7$	7.896	7.896	516	673	-0.001	0.080	0.080	0.947	0.304
$\mathbf{b}_8$	7.944	7.944	537	620	0.003	0.080	0.080	0.943	0.307
$\mathbf{b}_9$	7.988	7.988	582	533	0.009	0.081	0.081	0.945	0.313
$\mathbf{b}_{10}$	8.071	8.071	636	435	0.008	0.082	0.083	0.952	0.325
$\mathbf{b}_{11}$	8.811	8.811	788	391	0.001	0.091	0.091	0.946	0.351
$\mathbf{b}_{12}$	8.058	8.058	618	477	-0.004	0.081	0.082	0.946	0.317
$\mathbf{b}_{13}$	8.046	8.046	558	616	-0.001	0.078	0.078	0.951	0.303
$\mathbf{b}_{14}$	8.036	8.036	533	702	0.001	0.076	0.076	0.947	0.295
$\mathbf{b}_{15}$	7.741	7.741	515	675	-0.001	0.079	0.079	0.947	0.303
$\mathbf{b}_{16}$	7.337	7.337	476	616	-0.001	0.083	0.083	0.942	0.317
$\mathbf{b}_{17}$	7.081	7.081	445	572	-0.004	0.085	0.085	0.943	0.329
$\mathbf{b}_{18}$	7.087	7.087	438	562	-0.001	0.083	0.083	0.950	0.330
$\mathbf{b}_{19}$	7.332	7.332	451	585	0.003	0.085	0.085	0.938	0.325
$\mathbf{b}_{20}$	7.689	7.689	468	614	0.000	0.084	0.084	0.939	0.317
$\mathbf{b}_{21}$	8.101	8.101	482	641	-0.002	0.079	0.079	0.947	0.311
`Uniform`								0.935	0.484
`WBATE`					0.000	0.026	0.026	0.947	0.100
`LBATE`					0.118	0.051	0.129	0.989	0.497

Table SA-75: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	16.410	16.410	1693	3079	-0.003	0.039	0.039	0.947	0.285
$\mathbf{b}_2$	16.047	16.047	1671	2986	-0.003	0.040	0.040	0.948	0.290
$\mathbf{b}_3$	15.948	15.948	1694	2949	-0.003	0.041	0.041	0.956	0.292
$\mathbf{b}_4$	16.089	16.089	1775	2937	-0.001	0.043	0.043	0.959	0.292
$\mathbf{b}_5$	16.495	16.495	1926	2946	0.004	0.045	0.045	0.958	0.287
$\mathbf{b}_6$	17.158	17.158	2138	2971	0.010	0.045	0.046	0.955	0.280
$\mathbf{b}_7$	17.721	17.721	2326	2913	0.016	0.043	0.046	0.942	0.275
$\mathbf{b}_8$	18.359	18.359	2512	2837	0.018	0.041	0.044	0.941	0.270
$\mathbf{b}_9$	18.990	18.990	2663	2730	0.014	0.039	0.042	0.944	0.268
$\mathbf{b}_{10}$	19.803	19.803	2812	2645	0.007	0.039	0.040	0.951	0.269
$\mathbf{b}_{11}$	21.462	21.462	3055	2761	-0.002	0.040	0.040	0.954	0.284
$\mathbf{b}_{12}$	19.640	19.640	2792	2714	-0.009	0.039	0.040	0.953	0.263
$\mathbf{b}_{13}$	19.005	19.005	2650	2951	-0.015	0.040	0.042	0.954	0.260
$\mathbf{b}_{14}$	18.130	18.130	2396	3075	-0.014	0.042	0.044	0.950	0.266
$\mathbf{b}_{15}$	16.883	16.883	2036	3009	-0.007	0.045	0.045	0.957	0.282
$\mathbf{b}_{16}$	15.616	15.616	1700	2794	-0.000	0.044	0.044	0.953	0.300
$\mathbf{b}_{17}$	15.047	15.047	1543	2651	0.001	0.042	0.042	0.952	0.307
$\mathbf{b}_{18}$	15.155	15.155	1525	2644	0.001	0.041	0.041	0.953	0.305
$\mathbf{b}_{19}$	15.676	15.676	1553	2737	0.001	0.042	0.042	0.943	0.299
$\mathbf{b}_{20}$	16.282	16.282	1566	2825	-0.000	0.043	0.043	0.945	0.295
$\mathbf{b}_{21}$	16.774	16.774	1539	2833	-0.002	0.043	0.043	0.954	0.296
`Uniform`								0.947	0.429
`WBATE`					0.001	0.018	0.018	0.946	0.099
`LBATE`					0.038	0.033	0.051	0.991	0.461

Table SA-76: Intention-to-Treat Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.687	17.687	1868	3552	-0.003	0.041	0.041	0.952	0.294
$\mathbf{b}_2$	17.077	17.077	1823	3350	-0.003	0.041	0.041	0.955	0.297
$\mathbf{b}_3$	16.696	16.696	1810	3195	-0.002	0.041	0.041	0.956	0.297
$\mathbf{b}_4$	16.579	16.579	1851	3081	0.001	0.042	0.042	0.951	0.294
$\mathbf{b}_5$	16.641	16.641	1943	2966	0.006	0.043	0.043	0.950	0.290
$\mathbf{b}_6$	16.776	16.776	2057	2824	0.014	0.042	0.044	0.951	0.286
$\mathbf{b}_7$	16.861	16.861	2154	2632	0.019	0.041	0.046	0.951	0.284
$\mathbf{b}_8$	16.634	16.634	2170	2336	0.020	0.041	0.046	0.948	0.290
$\mathbf{b}_9$	16.538	16.538	2193	2064	0.017	0.041	0.044	0.948	0.296
$\mathbf{b}_{10}$	16.270	16.270	2159	1750	0.009	0.042	0.043	0.951	0.304
$\mathbf{b}_{11}$	16.701	16.701	2235	1582	-0.001	0.043	0.043	0.952	0.313
$\mathbf{b}_{12}$	16.495	16.495	2199	1901	-0.009	0.043	0.044	0.954	0.308
$\mathbf{b}_{13}$	16.770	16.770	2209	2319	-0.015	0.043	0.045	0.957	0.302
$\mathbf{b}_{14}$	17.120	17.120	2193	2745	-0.016	0.043	0.046	0.957	0.297
$\mathbf{b}_{15}$	16.936	16.936	2037	2983	-0.009	0.045	0.046	0.950	0.305
$\mathbf{b}_{16}$	16.613	16.613	1856	3091	-0.001	0.046	0.046	0.954	0.316
$\mathbf{b}_{17}$	16.555	16.555	1767	3158	0.001	0.046	0.046	0.948	0.326
$\mathbf{b}_{18}$	16.909	16.909	1769	3267	-0.000	0.045	0.045	0.951	0.328
$\mathbf{b}_{19}$	17.755	17.755	1827	3495	-0.002	0.045	0.045	0.947	0.324
$\mathbf{b}_{20}$	18.817	18.817	1881	3748	-0.004	0.045	0.046	0.946	0.321
$\mathbf{b}_{21}$	20.034	20.034	1917	3997	-0.006	0.045	0.046	0.951	0.319
`Uniform`								0.946	0.460
`WBATE`					0.001	0.018	0.018	0.949	0.103
`LBATE`					0.042	0.032	0.053	0.985	0.479

Table SA-77: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.460	17.460	1835	3461	-0.003	0.041	0.041	0.952	0.297
$\mathbf{b}_2$	16.507	16.507	1732	3136	-0.002	0.042	0.042	0.954	0.304
$\mathbf{b}_3$	15.414	15.414	1590	2748	-0.001	0.043	0.043	0.953	0.317
$\mathbf{b}_4$	13.916	13.916	1371	2234	0.000	0.047	0.047	0.945	0.343
$\mathbf{b}_5$	11.999	11.999	1082	1642	0.001	0.053	0.053	0.939	0.392
$\mathbf{b}_6$	10.000	10.000	788	1113	0.002	0.062	0.062	0.940	0.466
$\mathbf{b}_7$	8.061	8.061	529	697	0.001	0.078	0.078	0.938	0.575
$\mathbf{b}_8$	7.300	7.300	450	532	0.002	0.088	0.088	0.939	0.640
$\mathbf{b}_9$	7.245	7.245	476	447	0.009	0.090	0.090	0.938	0.657
$\mathbf{b}_{10}$	7.128	7.128	500	343	0.008	0.091	0.091	0.941	0.678
$\mathbf{b}_{11}$	7.317	7.317	563	261	-0.000	0.094	0.094	0.930	0.701
$\mathbf{b}_{12}$	7.227	7.227	497	390	-0.005	0.092	0.092	0.942	0.681
$\mathbf{b}_{13}$	7.348	7.348	461	524	-0.002	0.089	0.089	0.943	0.665
$\mathbf{b}_{14}$	8.409	8.409	573	765	-0.001	0.078	0.078	0.942	0.583
$\mathbf{b}_{15}$	11.200	11.200	962	1418	-0.000	0.061	0.061	0.942	0.447
$\mathbf{b}_{16}$	13.919	13.919	1371	2234	0.000	0.050	0.050	0.947	0.369
$\mathbf{b}_{17}$	15.793	15.793	1640	2883	0.001	0.047	0.047	0.946	0.338
$\mathbf{b}_{18}$	16.766	16.766	1748	3211	-0.000	0.046	0.046	0.951	0.330
$\mathbf{b}_{19}$	17.739	17.739	1825	3488	-0.002	0.045	0.045	0.948	0.325
$\mathbf{b}_{20}$	18.816	18.816	1881	3747	-0.004	0.045	0.046	0.946	0.321
$\mathbf{b}_{21}$	20.034	20.034	1917	3997	-0.006	0.045	0.046	0.951	0.319
`Uniform`								0.917	0.711
`WBATE`					-0.000	0.023	0.023	0.944	0.128
`LBATE`					0.104	0.057	0.119	0.969	0.899

Table SA-78: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.749	7.749	513	675	-0.003	0.086	0.086	0.949	0.337
$\mathbf{b}_2$	7.482	7.482	490	638	-0.001	0.086	0.086	0.954	0.340
$\mathbf{b}_3$	7.315	7.315	474	614	-0.001	0.086	0.086	0.955	0.342
$\mathbf{b}_4$	7.264	7.264	467	604	-0.000	0.089	0.089	0.943	0.339
$\mathbf{b}_5$	7.290	7.290	465	602	0.001	0.088	0.088	0.943	0.335
$\mathbf{b}_6$	7.350	7.350	463	600	0.001	0.086	0.086	0.948	0.331
$\mathbf{b}_7$	7.387	7.387	455	589	0.000	0.086	0.086	0.951	0.330
$\mathbf{b}_8$	7.287	7.287	449	530	0.002	0.088	0.088	0.948	0.336
$\mathbf{b}_9$	7.245	7.245	476	447	0.009	0.090	0.090	0.945	0.343
$\mathbf{b}_{10}$	7.128	7.128	500	343	0.008	0.091	0.091	0.948	0.352
$\mathbf{b}_{11}$	7.317	7.317	563	261	-0.000	0.094	0.094	0.948	0.365
$\mathbf{b}_{12}$	7.227	7.227	497	390	-0.005	0.092	0.092	0.951	0.356
$\mathbf{b}_{13}$	7.347	7.347	461	523	-0.002	0.089	0.089	0.950	0.349
$\mathbf{b}_{14}$	7.501	7.501	469	610	-0.000	0.088	0.088	0.952	0.343
$\mathbf{b}_{15}$	7.420	7.420	477	619	0.000	0.092	0.092	0.942	0.349
$\mathbf{b}_{16}$	7.278	7.278	468	606	0.004	0.094	0.094	0.946	0.361
$\mathbf{b}_{17}$	7.253	7.253	465	601	0.004	0.097	0.097	0.942	0.370
$\mathbf{b}_{18}$	7.408	7.408	475	617	0.000	0.096	0.096	0.945	0.372
$\mathbf{b}_{19}$	7.779	7.779	502	661	-0.002	0.094	0.094	0.949	0.367
$\mathbf{b}_{20}$	8.244	8.244	530	710	-0.001	0.093	0.093	0.948	0.362
$\mathbf{b}_{21}$	8.777	8.777	556	759	-0.000	0.092	0.092	0.950	0.358
`Uniform`								0.933	0.535
`WBATE`					0.001	0.027	0.027	0.950	0.106
`LBATE`					0.137	0.057	0.148	0.983	0.553

Table SA-79: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	16.600	16.600	1719	3154	-0.004	0.045	0.046	0.953	0.313
$\mathbf{b}_2$	16.166	16.166	1690	3035	-0.004	0.045	0.045	0.959	0.315
$\mathbf{b}_3$	15.882	15.882	1686	2931	-0.004	0.045	0.046	0.960	0.315
$\mathbf{b}_4$	15.852	15.852	1738	2864	-0.002	0.047	0.047	0.955	0.313
$\mathbf{b}_5$	16.025	16.025	1840	2807	0.003	0.049	0.049	0.952	0.308
$\mathbf{b}_6$	16.350	16.350	1980	2738	0.009	0.049	0.049	0.951	0.301
$\mathbf{b}_7$	16.274	16.274	2037	2515	0.014	0.049	0.051	0.949	0.302
$\mathbf{b}_8$	16.244	16.244	2091	2279	0.016	0.048	0.051	0.947	0.304
$\mathbf{b}_9$	16.123	16.123	2107	2009	0.014	0.047	0.049	0.951	0.311
$\mathbf{b}_{10}$	15.915	15.915	2087	1714	0.007	0.047	0.047	0.954	0.318
$\mathbf{b}_{11}$	16.264	16.264	2148	1536	-0.002	0.047	0.047	0.955	0.329
$\mathbf{b}_{12}$	16.237	16.237	2145	1881	-0.009	0.047	0.048	0.953	0.319
$\mathbf{b}_{13}$	16.776	16.776	2209	2356	-0.013	0.047	0.049	0.960	0.308
$\mathbf{b}_{14}$	16.977	16.977	2166	2747	-0.012	0.049	0.051	0.957	0.308
$\mathbf{b}_{15}$	16.647	16.647	1996	2940	-0.005	0.053	0.053	0.956	0.321
$\mathbf{b}_{16}$	15.901	15.901	1757	2891	0.003	0.052	0.052	0.958	0.340
$\mathbf{b}_{17}$	15.588	15.588	1630	2851	0.005	0.052	0.052	0.950	0.351
$\mathbf{b}_{18}$	15.786	15.786	1612	2884	0.003	0.051	0.051	0.950	0.353
$\mathbf{b}_{19}$	16.349	16.349	1641	2994	0.001	0.051	0.052	0.952	0.352
$\mathbf{b}_{20}$	17.026	17.026	1658	3109	0.000	0.053	0.053	0.950	0.354
$\mathbf{b}_{21}$	17.757	17.757	1651	3197	0.000	0.054	0.054	0.956	0.359
`Uniform`								0.943	0.491
`WBATE`					0.001	0.020	0.020	0.947	0.106
`LBATE`					0.054	0.041	0.068	0.984	0.551

Table SA-80: Intention-to-Treat Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.940	18.940	2038	4030	-0.001	0.028	0.028	0.947	0.214
$\mathbf{b}_2$	18.560	18.560	2046	3894	-0.001	0.029	0.029	0.945	0.217
$\mathbf{b}_3$	18.341	18.341	2087	3764	-0.001	0.029	0.029	0.944	0.219
$\mathbf{b}_4$	18.247	18.247	2164	3620	-0.001	0.030	0.030	0.945	0.220
$\mathbf{b}_5$	18.265	18.265	2269	3460	-0.002	0.030	0.030	0.942	0.222
$\mathbf{b}_6$	18.404	18.404	2394	3295	-0.002	0.030	0.030	0.944	0.223
$\mathbf{b}_7$	18.492	18.492	2494	3084	-0.001	0.030	0.030	0.944	0.226
$\mathbf{b}_8$	18.409	18.409	2536	2801	-0.001	0.032	0.032	0.948	0.233
$\mathbf{b}_9$	18.445	18.445	2577	2534	0.000	0.033	0.033	0.945	0.241
$\mathbf{b}_{10}$	18.570	18.570	2606	2280	0.000	0.036	0.036	0.945	0.261
$\mathbf{b}_{11}$	20.183	20.183	2867	2376	-0.001	0.038	0.038	0.944	0.298
$\mathbf{b}_{12}$	18.471	18.471	2589	2371	-0.004	0.035	0.035	0.945	0.251
$\mathbf{b}_{13}$	18.461	18.461	2555	2759	-0.006	0.031	0.032	0.943	0.233
$\mathbf{b}_{14}$	18.567	18.567	2494	3154	-0.007	0.029	0.030	0.943	0.223
$\mathbf{b}_{15}$	18.485	18.485	2354	3449	-0.005	0.029	0.029	0.947	0.218
$\mathbf{b}_{16}$	18.323	18.323	2178	3645	-0.002	0.029	0.029	0.948	0.217
$\mathbf{b}_{17}$	18.304	18.304	2048	3779	-0.001	0.029	0.029	0.947	0.217
$\mathbf{b}_{18}$	18.620	18.620	2006	3912	0.000	0.029	0.029	0.945	0.214
$\mathbf{b}_{19}$	19.300	19.300	2021	4091	0.001	0.028	0.028	0.948	0.209
$\mathbf{b}_{20}$	20.308	20.308	2057	4323	0.001	0.027	0.027	0.945	0.202
$\mathbf{b}_{21}$	21.402	21.402	2067	4519	0.002	0.026	0.026	0.949	0.197
`Uniform`								0.935	0.339
`WBATE`					-0.001	0.015	0.015	0.950	0.086
`LBATE`					0.010	0.023	0.025	0.993	0.329

Table SA-81: First-Stage Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.446	18.446	1967	3834	-0.001	0.029	0.029	0.947	0.218
$\mathbf{b}_2$	17.301	17.301	1843	3422	-0.001	0.030	0.030	0.947	0.230
$\mathbf{b}_3$	15.833	15.833	1650	2887	-0.001	0.033	0.033	0.943	0.249
$\mathbf{b}_4$	13.993	13.993	1382	2255	-0.000	0.038	0.038	0.942	0.282
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.001	0.044	0.044	0.940	0.331
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.001	0.053	0.053	0.948	0.401
$\mathbf{b}_7$	8.375	8.375	570	751	-0.002	0.064	0.064	0.941	0.487
$\mathbf{b}_8$	8.066	8.066	554	636	-0.002	0.070	0.070	0.931	0.520
$\mathbf{b}_9$	8.081	8.081	595	543	-0.000	0.073	0.073	0.932	0.539
$\mathbf{b}_{10}$	8.136	8.136	646	441	-0.001	0.078	0.078	0.931	0.564
$\mathbf{b}_{11}$	8.842	8.842	794	394	-0.001	0.093	0.093	0.916	0.697
$\mathbf{b}_{12}$	8.092	8.092	623	480	-0.004	0.075	0.075	0.931	0.554
$\mathbf{b}_{13}$	8.088	8.088	564	622	-0.002	0.070	0.070	0.933	0.521
$\mathbf{b}_{14}$	8.563	8.563	593	793	-0.002	0.062	0.063	0.939	0.471
$\mathbf{b}_{15}$	11.200	11.200	961	1419	-0.000	0.047	0.047	0.936	0.353
$\mathbf{b}_{16}$	13.993	13.993	1381	2257	-0.000	0.037	0.037	0.942	0.279
$\mathbf{b}_{17}$	16.467	16.467	1737	3117	-0.000	0.031	0.031	0.949	0.237
$\mathbf{b}_{18}$	18.090	18.090	1927	3704	0.000	0.029	0.029	0.946	0.219
$\mathbf{b}_{19}$	19.259	19.259	2016	4074	0.001	0.028	0.028	0.948	0.209
$\mathbf{b}_{20}$	20.308	20.308	2057	4323	0.001	0.027	0.027	0.945	0.202
$\mathbf{b}_{21}$	21.402	21.402	2067	4519	0.002	0.026	0.026	0.949	0.197
`Uniform`								0.881	0.565
`WBATE`					-0.001	0.019	0.019	0.951	0.111
`LBATE`					0.037	0.037	0.052	0.976	0.665

Table SA-82: First-Stage Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.298	8.298	578	774	-0.001	0.066	0.066	0.948	0.253
$\mathbf{b}_2$	8.131	8.131	566	754	-0.001	0.066	0.066	0.944	0.256
$\mathbf{b}_3$	8.035	8.035	559	741	-0.001	0.067	0.067	0.940	0.258
$\mathbf{b}_4$	7.994	7.994	553	733	0.000	0.068	0.068	0.938	0.259
$\mathbf{b}_5$	8.002	8.002	549	726	-0.000	0.068	0.068	0.942	0.259
$\mathbf{b}_6$	8.063	8.063	546	723	-0.002	0.066	0.066	0.947	0.259
$\mathbf{b}_7$	8.101	8.101	540	706	-0.003	0.066	0.066	0.949	0.261
$\mathbf{b}_8$	8.065	8.065	554	636	-0.002	0.070	0.070	0.943	0.270
$\mathbf{b}_9$	8.081	8.081	595	543	-0.000	0.073	0.073	0.942	0.280
$\mathbf{b}_{10}$	8.136	8.136	646	441	-0.001	0.078	0.078	0.942	0.302
$\mathbf{b}_{11}$	8.842	8.842	794	394	-0.001	0.093	0.093	0.940	0.359
$\mathbf{b}_{12}$	8.092	8.092	623	480	-0.004	0.075	0.075	0.941	0.290
$\mathbf{b}_{13}$	8.088	8.088	564	622	-0.002	0.070	0.070	0.946	0.270
$\mathbf{b}_{14}$	8.134	8.134	544	719	-0.002	0.067	0.067	0.940	0.258
$\mathbf{b}_{15}$	8.098	8.098	556	739	-0.000	0.067	0.067	0.939	0.254
$\mathbf{b}_{16}$	8.028	8.028	556	738	0.000	0.066	0.066	0.944	0.255
$\mathbf{b}_{17}$	8.019	8.019	555	737	-0.000	0.065	0.065	0.948	0.255
$\mathbf{b}_{18}$	8.157	8.157	562	751	-0.002	0.066	0.066	0.940	0.252
$\mathbf{b}_{19}$	8.456	8.456	580	782	-0.002	0.064	0.064	0.946	0.246
$\mathbf{b}_{20}$	8.897	8.897	604	827	-0.001	0.062	0.062	0.947	0.239
$\mathbf{b}_{21}$	9.376	9.376	621	865	0.000	0.061	0.061	0.943	0.233
`Uniform`								0.920	0.404
`WBATE`					-0.001	0.022	0.022	0.956	0.088
`LBATE`					0.071	0.042	0.083	0.989	0.394

Table SA-83: First-Stage Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.446	18.446	1976	3846	-0.001	0.029	0.029	0.946	0.222
$\mathbf{b}_2$	18.337	18.337	2020	3819	-0.001	0.030	0.030	0.945	0.222
$\mathbf{b}_3$	18.348	18.348	2101	3775	-0.001	0.030	0.030	0.946	0.223
$\mathbf{b}_4$	18.488	18.488	2219	3716	-0.001	0.030	0.030	0.945	0.222
$\mathbf{b}_5$	18.737	18.737	2365	3637	-0.002	0.030	0.030	0.944	0.222
$\mathbf{b}_6$	18.966	18.966	2501	3501	-0.002	0.030	0.030	0.945	0.222
$\mathbf{b}_7$	19.098	19.098	2604	3300	-0.001	0.031	0.031	0.946	0.225
$\mathbf{b}_8$	19.288	19.288	2695	3089	-0.000	0.032	0.032	0.947	0.228
$\mathbf{b}_9$	19.613	19.613	2782	2892	0.000	0.032	0.032	0.945	0.233
$\mathbf{b}_{10}$	20.205	20.205	2884	2747	0.000	0.034	0.034	0.942	0.247
$\mathbf{b}_{11}$	21.806	21.806	3110	2852	-0.001	0.037	0.037	0.943	0.282
$\mathbf{b}_{12}$	19.941	19.941	2845	2797	-0.004	0.033	0.034	0.947	0.239
$\mathbf{b}_{13}$	19.520	19.520	2749	3093	-0.006	0.031	0.031	0.941	0.226
$\mathbf{b}_{14}$	19.298	19.298	2631	3407	-0.007	0.029	0.030	0.943	0.220
$\mathbf{b}_{15}$	19.009	19.009	2456	3648	-0.005	0.030	0.030	0.944	0.219
$\mathbf{b}_{16}$	18.515	18.515	2227	3730	-0.003	0.030	0.030	0.947	0.221
$\mathbf{b}_{17}$	18.115	18.115	2033	3722	-0.001	0.030	0.030	0.951	0.224
$\mathbf{b}_{18}$	18.047	18.047	1931	3705	-0.000	0.030	0.030	0.943	0.223
$\mathbf{b}_{19}$	18.280	18.280	1892	3705	0.000	0.030	0.030	0.947	0.222
$\mathbf{b}_{20}$	18.825	18.825	1878	3762	0.001	0.029	0.029	0.945	0.220
$\mathbf{b}_{21}$	19.374	19.374	1839	3766	0.002	0.029	0.030	0.946	0.220
`Uniform`								0.930	0.341
`WBATE`					-0.002	0.015	0.015	0.950	0.086
`LBATE`					0.012	0.025	0.028	0.992	0.350

Table SA-84: First-Stage Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.488	19.488	2114	4244	-0.001	0.027	0.027	0.949	0.208
$\mathbf{b}_2$	18.846	18.846	2089	4002	-0.001	0.028	0.028	0.949	0.213
$\mathbf{b}_3$	18.345	18.345	2087	3766	-0.001	0.029	0.029	0.950	0.218
$\mathbf{b}_4$	17.972	17.972	2110	3530	-0.001	0.030	0.030	0.951	0.223
$\mathbf{b}_5$	17.684	17.684	2151	3279	-0.001	0.031	0.031	0.948	0.228
$\mathbf{b}_6$	17.475	17.475	2202	3021	-0.001	0.031	0.032	0.943	0.234
$\mathbf{b}_7$	17.293	17.293	2244	2747	-0.001	0.033	0.033	0.945	0.242
$\mathbf{b}_8$	16.916	16.916	2229	2406	-0.001	0.035	0.035	0.943	0.253
$\mathbf{b}_9$	16.732	16.732	2233	2109	0.000	0.037	0.037	0.944	0.265
$\mathbf{b}_{10}$	16.448	16.448	2194	1787	0.000	0.041	0.041	0.939	0.293
$\mathbf{b}_{11}$	16.857	16.857	2265	1614	-0.000	0.048	0.048	0.941	0.360
$\mathbf{b}_{12}$	16.613	16.613	2223	1927	-0.004	0.039	0.039	0.942	0.277
$\mathbf{b}_{13}$	16.893	16.893	2234	2351	-0.005	0.034	0.034	0.945	0.254
$\mathbf{b}_{14}$	17.334	17.334	2239	2803	-0.005	0.032	0.032	0.946	0.238
$\mathbf{b}_{15}$	17.565	17.565	2165	3168	-0.004	0.031	0.031	0.941	0.229
$\mathbf{b}_{16}$	17.929	17.929	2101	3515	-0.001	0.030	0.030	0.939	0.221
$\mathbf{b}_{17}$	18.505	18.505	2078	3852	0.000	0.029	0.029	0.944	0.213
$\mathbf{b}_{18}$	19.329	19.329	2103	4188	0.001	0.028	0.028	0.947	0.205
$\mathbf{b}_{19}$	20.492	20.492	2167	4569	0.002	0.026	0.026	0.949	0.196
$\mathbf{b}_{20}$	21.968	21.968	2239	4998	0.003	0.025	0.025	0.946	0.186
$\mathbf{b}_{21}$	23.598	23.598	2285	5410	0.004	0.024	0.024	0.945	0.178
`Uniform`								0.927	0.353
`WBATE`					-0.001	0.016	0.016	0.948	0.088
`LBATE`					0.011	0.022	0.024	0.993	0.347

Table SA-85: First-Stage Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.811	18.811	2015	3976	-0.001	0.028	0.028	0.950	0.214
$\mathbf{b}_2$	17.441	17.441	1863	3475	-0.001	0.030	0.030	0.951	0.228
$\mathbf{b}_3$	15.849	15.849	1652	2894	-0.000	0.032	0.032	0.953	0.249
$\mathbf{b}_4$	13.990	13.990	1381	2255	-0.000	0.037	0.037	0.946	0.282
$\mathbf{b}_5$	12.000	12.000	1081	1641	-0.000	0.043	0.043	0.945	0.331
$\mathbf{b}_6$	10.000	10.000	788	1112	-0.001	0.052	0.052	0.941	0.401
$\mathbf{b}_7$	8.099	8.099	534	703	-0.000	0.067	0.066	0.940	0.503
$\mathbf{b}_8$	7.421	7.421	465	547	0.000	0.075	0.075	0.933	0.566
$\mathbf{b}_9$	7.330	7.330	487	456	-0.002	0.079	0.079	0.935	0.592
$\mathbf{b}_{10}$	7.206	7.206	510	349	-0.002	0.088	0.088	0.922	0.637
$\mathbf{b}_{11}$	7.385	7.385	572	266	-0.001	0.115	0.115	0.904	0.834
$\mathbf{b}_{12}$	7.278	7.278	504	395	-0.003	0.083	0.083	0.926	0.614
$\mathbf{b}_{13}$	7.402	7.402	469	531	-0.000	0.075	0.075	0.934	0.568
$\mathbf{b}_{14}$	8.414	8.414	574	766	0.001	0.063	0.063	0.934	0.478
$\mathbf{b}_{15}$	11.200	11.200	962	1419	0.000	0.047	0.047	0.940	0.353
$\mathbf{b}_{16}$	13.989	13.989	1381	2256	0.000	0.037	0.037	0.941	0.279
$\mathbf{b}_{17}$	16.557	16.557	1750	3151	0.001	0.032	0.032	0.939	0.236
$\mathbf{b}_{18}$	18.615	18.615	1998	3908	0.001	0.028	0.028	0.946	0.212
$\mathbf{b}_{19}$	20.332	20.332	2147	4503	0.002	0.026	0.026	0.949	0.197
$\mathbf{b}_{20}$	21.947	21.947	2236	4989	0.003	0.025	0.025	0.946	0.186
$\mathbf{b}_{21}$	23.593	23.593	2284	5408	0.004	0.024	0.024	0.945	0.178
`Uniform`								0.867	0.592
`WBATE`					0.000	0.020	0.020	0.945	0.114
`LBATE`					0.046	0.045	0.064	0.970	0.759

Table SA-86: First-Stage Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.538	8.538	607	820	-0.001	0.063	0.063	0.950	0.245
$\mathbf{b}_2$	8.257	8.257	582	778	-0.000	0.064	0.064	0.951	0.252
$\mathbf{b}_3$	8.037	8.037	559	741	0.000	0.067	0.067	0.946	0.257
$\mathbf{b}_4$	7.874	7.874	538	709	-0.000	0.069	0.069	0.943	0.262
$\mathbf{b}_5$	7.747	7.747	518	679	-0.001	0.069	0.069	0.945	0.268
$\mathbf{b}_6$	7.656	7.656	498	650	-0.001	0.070	0.070	0.946	0.273
$\mathbf{b}_7$	7.576	7.576	477	618	-0.000	0.072	0.072	0.941	0.280
$\mathbf{b}_8$	7.411	7.411	465	546	0.000	0.075	0.075	0.947	0.293
$\mathbf{b}_9$	7.330	7.330	487	456	-0.002	0.079	0.079	0.944	0.308
$\mathbf{b}_{10}$	7.206	7.206	510	349	-0.002	0.088	0.088	0.936	0.338
$\mathbf{b}_{11}$	7.385	7.385	572	266	-0.001	0.115	0.115	0.929	0.434
$\mathbf{b}_{12}$	7.278	7.278	504	395	-0.003	0.083	0.083	0.944	0.321
$\mathbf{b}_{13}$	7.401	7.401	468	530	-0.000	0.075	0.075	0.945	0.295
$\mathbf{b}_{14}$	7.594	7.594	480	625	0.001	0.071	0.071	0.943	0.276
$\mathbf{b}_{15}$	7.695	7.695	508	665	-0.000	0.070	0.070	0.942	0.268
$\mathbf{b}_{16}$	7.855	7.855	536	705	0.001	0.067	0.067	0.943	0.260
$\mathbf{b}_{17}$	8.107	8.107	566	752	0.002	0.066	0.066	0.943	0.251
$\mathbf{b}_{18}$	8.468	8.468	601	809	0.000	0.063	0.063	0.944	0.242
$\mathbf{b}_{19}$	8.978	8.978	644	883	0.000	0.059	0.059	0.948	0.231
$\mathbf{b}_{20}$	9.624	9.624	692	970	0.002	0.057	0.057	0.944	0.220
$\mathbf{b}_{21}$	10.338	10.338	734	1056	0.002	0.055	0.055	0.940	0.210
`Uniform`								0.909	0.420
`WBATE`					-0.000	0.023	0.023	0.949	0.090
`LBATE`					0.076	0.045	0.088	0.984	0.420

Table SA-87: First-Stage Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.343	18.343	1960	3818	-0.001	0.029	0.029	0.947	0.224
$\mathbf{b}_2$	18.051	18.051	1977	3719	-0.001	0.030	0.030	0.950	0.227
$\mathbf{b}_3$	17.855	17.855	2018	3608	-0.001	0.031	0.031	0.948	0.229
$\mathbf{b}_4$	17.785	17.785	2090	3485	-0.001	0.031	0.031	0.947	0.231
$\mathbf{b}_5$	17.710	17.710	2166	3313	-0.001	0.032	0.032	0.948	0.234
$\mathbf{b}_6$	17.660	17.660	2243	3107	-0.002	0.032	0.033	0.942	0.238
$\mathbf{b}_7$	17.157	17.157	2217	2745	-0.001	0.034	0.034	0.943	0.250
$\mathbf{b}_8$	16.803	16.803	2205	2415	-0.001	0.036	0.036	0.945	0.262
$\mathbf{b}_9$	16.462	16.462	2176	2085	-0.000	0.039	0.039	0.944	0.277
$\mathbf{b}_{10}$	16.219	16.219	2146	1778	0.000	0.043	0.043	0.936	0.304
$\mathbf{b}_{11}$	16.492	16.492	2191	1581	-0.001	0.051	0.051	0.940	0.379
$\mathbf{b}_{12}$	16.520	16.520	2200	1947	-0.004	0.040	0.040	0.942	0.285
$\mathbf{b}_{13}$	17.243	17.243	2302	2481	-0.006	0.035	0.035	0.945	0.256
$\mathbf{b}_{14}$	17.877	17.877	2349	2999	-0.005	0.032	0.033	0.945	0.238
$\mathbf{b}_{15}$	18.466	18.466	2354	3483	-0.004	0.031	0.031	0.939	0.226
$\mathbf{b}_{16}$	18.673	18.673	2263	3788	-0.002	0.031	0.031	0.939	0.220
$\mathbf{b}_{17}$	18.855	18.855	2164	3992	-0.000	0.030	0.030	0.941	0.217
$\mathbf{b}_{18}$	19.160	19.160	2095	4145	0.001	0.029	0.029	0.945	0.213
$\mathbf{b}_{19}$	19.705	19.705	2067	4290	0.002	0.028	0.028	0.948	0.210
$\mathbf{b}_{20}$	20.543	20.543	2067	4467	0.003	0.027	0.027	0.945	0.205
$\mathbf{b}_{21}$	21.462	21.462	2050	4606	0.004	0.028	0.028	0.947	0.202
`Uniform`								0.923	0.367
`WBATE`					-0.001	0.016	0.016	0.946	0.089
`LBATE`					0.014	0.025	0.029	0.992	0.383

Table SA-88: First-Stage Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.240	17.240	1807	3381	-0.003	0.031	0.031	0.947	0.235
$\mathbf{b}_2$	16.756	16.756	1775	3231	-0.003	0.032	0.032	0.951	0.240
$\mathbf{b}_3$	16.586	16.586	1790	3156	-0.003	0.032	0.032	0.947	0.242
$\mathbf{b}_4$	16.714	16.714	1873	3125	-0.002	0.032	0.032	0.946	0.241
$\mathbf{b}_5$	17.111	17.111	2036	3110	0.001	0.032	0.032	0.947	0.237
$\mathbf{b}_6$	17.598	17.598	2226	3064	0.004	0.031	0.032	0.945	0.233
$\mathbf{b}_7$	18.024	18.024	2395	2954	0.006	0.031	0.032	0.948	0.232
$\mathbf{b}_8$	18.133	18.133	2479	2728	0.007	0.032	0.032	0.946	0.235
$\mathbf{b}_9$	18.233	18.233	2535	2481	0.005	0.033	0.034	0.945	0.242
$\mathbf{b}_{10}$	18.423	18.423	2578	2245	0.001	0.036	0.036	0.949	0.260
$\mathbf{b}_{11}$	20.110	20.110	2855	2358	-0.005	0.038	0.039	0.948	0.296
$\mathbf{b}_{12}$	18.393	18.393	2574	2351	-0.007	0.035	0.035	0.947	0.248
$\mathbf{b}_{13}$	18.366	18.366	2537	2735	-0.007	0.031	0.032	0.944	0.231
$\mathbf{b}_{14}$	18.343	18.343	2448	3092	-0.006	0.030	0.031	0.937	0.223
$\mathbf{b}_{15}$	17.668	17.668	2187	3205	-0.004	0.030	0.030	0.947	0.226
$\mathbf{b}_{16}$	16.748	16.748	1880	3135	-0.002	0.031	0.031	0.943	0.234
$\mathbf{b}_{17}$	16.162	16.162	1707	3017	-0.002	0.032	0.032	0.946	0.242
$\mathbf{b}_{18}$	16.176	16.176	1668	2995	-0.001	0.032	0.032	0.945	0.242
$\mathbf{b}_{19}$	16.736	16.736	1694	3114	-0.001	0.031	0.031	0.944	0.237
$\mathbf{b}_{20}$	17.551	17.551	1727	3273	-0.001	0.030	0.030	0.950	0.231
$\mathbf{b}_{21}$	18.491	18.491	1744	3422	-0.001	0.030	0.030	0.948	0.226
`Uniform`								0.926	0.360
`WBATE`					-0.001	0.015	0.016	0.948	0.088
`LBATE`					0.016	0.025	0.030	0.986	0.357

Table SA-89: First-Stage Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.072	17.072	1783	3314	-0.003	0.031	0.031	0.947	0.237
$\mathbf{b}_2$	16.314	16.314	1705	3064	-0.003	0.032	0.032	0.951	0.245
$\mathbf{b}_3$	15.415	15.415	1589	2746	-0.002	0.034	0.034	0.947	0.257
$\mathbf{b}_4$	13.947	13.947	1374	2243	-0.002	0.037	0.037	0.943	0.284
$\mathbf{b}_5$	12.000	12.000	1080	1642	-0.001	0.044	0.044	0.942	0.331
$\mathbf{b}_6$	10.000	10.000	787	1114	-0.000	0.053	0.053	0.943	0.400
$\mathbf{b}_7$	8.292	8.292	559	737	-0.000	0.066	0.065	0.935	0.489
$\mathbf{b}_8$	7.946	7.946	537	620	0.001	0.070	0.070	0.934	0.526
$\mathbf{b}_9$	7.988	7.988	582	533	0.003	0.072	0.072	0.928	0.540
$\mathbf{b}_{10}$	8.071	8.071	636	435	0.003	0.076	0.077	0.931	0.563
$\mathbf{b}_{11}$	8.811	8.811	788	391	-0.002	0.091	0.091	0.915	0.691
$\mathbf{b}_{12}$	8.058	8.058	618	477	-0.002	0.073	0.073	0.932	0.548
$\mathbf{b}_{13}$	8.047	8.047	558	616	-0.000	0.069	0.069	0.930	0.514
$\mathbf{b}_{14}$	8.537	8.537	590	788	0.000	0.062	0.062	0.934	0.465
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.046	0.046	0.945	0.348
$\mathbf{b}_{16}$	13.954	13.954	1376	2245	-0.002	0.037	0.037	0.939	0.276
$\mathbf{b}_{17}$	15.604	15.604	1614	2816	-0.002	0.033	0.033	0.944	0.248
$\mathbf{b}_{18}$	16.095	16.095	1656	2963	-0.001	0.032	0.032	0.945	0.243
$\mathbf{b}_{19}$	16.730	16.730	1693	3111	-0.001	0.031	0.031	0.944	0.237
$\mathbf{b}_{20}$	17.551	17.551	1727	3273	-0.001	0.030	0.030	0.950	0.231
$\mathbf{b}_{21}$	18.491	18.491	1744	3422	-0.001	0.030	0.030	0.948	0.226
`Uniform`								0.879	0.576
`WBATE`					-0.001	0.019	0.019	0.948	0.112
`LBATE`					0.047	0.040	0.062	0.962	0.675

Table SA-90: First-Stage Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.553	7.553	489	640	0.001	0.072	0.072	0.942	0.279
$\mathbf{b}_2$	7.341	7.341	472	613	0.001	0.074	0.074	0.944	0.284
$\mathbf{b}_3$	7.267	7.267	467	605	0.000	0.074	0.074	0.941	0.286
$\mathbf{b}_4$	7.322	7.322	473	615	-0.001	0.073	0.073	0.946	0.284
$\mathbf{b}_5$	7.496	7.496	489	639	0.000	0.072	0.072	0.943	0.278
$\mathbf{b}_6$	7.710	7.710	505	664	-0.000	0.070	0.070	0.944	0.271
$\mathbf{b}_7$	7.896	7.896	516	673	-0.000	0.070	0.070	0.943	0.267
$\mathbf{b}_8$	7.944	7.944	537	620	0.001	0.070	0.070	0.942	0.273
$\mathbf{b}_9$	7.988	7.988	582	533	0.003	0.072	0.072	0.945	0.281
$\mathbf{b}_{10}$	8.071	8.071	636	435	0.003	0.076	0.077	0.946	0.301
$\mathbf{b}_{11}$	8.811	8.811	788	391	-0.002	0.091	0.091	0.939	0.356
$\mathbf{b}_{12}$	8.058	8.058	618	477	-0.002	0.073	0.073	0.949	0.287
$\mathbf{b}_{13}$	8.046	8.046	558	616	-0.000	0.069	0.069	0.943	0.268
$\mathbf{b}_{14}$	8.036	8.036	533	702	0.000	0.067	0.067	0.940	0.257
$\mathbf{b}_{15}$	7.741	7.741	515	675	-0.001	0.068	0.068	0.947	0.264
$\mathbf{b}_{16}$	7.337	7.337	476	616	-0.001	0.073	0.073	0.937	0.276
$\mathbf{b}_{17}$	7.081	7.081	445	572	-0.001	0.074	0.074	0.941	0.285
$\mathbf{b}_{18}$	7.087	7.087	438	562	0.001	0.075	0.075	0.938	0.286
$\mathbf{b}_{19}$	7.332	7.332	451	585	0.002	0.072	0.072	0.942	0.280
$\mathbf{b}_{20}$	7.689	7.689	468	614	0.000	0.071	0.071	0.943	0.273
$\mathbf{b}_{21}$	8.101	8.101	482	641	-0.001	0.069	0.069	0.944	0.267
`Uniform`								0.909	0.429
`WBATE`					0.000	0.023	0.023	0.946	0.090
`LBATE`					0.088	0.046	0.099	0.980	0.419

Table SA-91: First-Stage Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	16.410	16.410	1693	3079	-0.003	0.033	0.033	0.945	0.248
$\mathbf{b}_2$	16.047	16.047	1671	2986	-0.003	0.033	0.034	0.950	0.253
$\mathbf{b}_3$	15.948	15.948	1694	2949	-0.002	0.034	0.034	0.945	0.255
$\mathbf{b}_4$	16.089	16.089	1775	2937	-0.001	0.034	0.034	0.947	0.255
$\mathbf{b}_5$	16.495	16.495	1926	2946	0.001	0.034	0.034	0.946	0.252
$\mathbf{b}_6$	17.158	17.158	2138	2971	0.003	0.034	0.034	0.947	0.246
$\mathbf{b}_7$	17.721	17.721	2326	2913	0.005	0.033	0.034	0.945	0.242
$\mathbf{b}_8$	18.359	18.359	2512	2837	0.006	0.033	0.033	0.947	0.239
$\mathbf{b}_9$	18.990	18.990	2663	2730	0.005	0.033	0.033	0.945	0.239
$\mathbf{b}_{10}$	19.803	19.803	2812	2645	0.000	0.035	0.035	0.947	0.249
$\mathbf{b}_{11}$	21.462	21.462	3055	2761	-0.006	0.037	0.038	0.944	0.284
$\mathbf{b}_{12}$	19.640	19.640	2792	2714	-0.008	0.033	0.034	0.948	0.239
$\mathbf{b}_{13}$	19.005	19.005	2650	2951	-0.008	0.031	0.032	0.944	0.229
$\mathbf{b}_{14}$	18.130	18.130	2396	3075	-0.006	0.032	0.032	0.937	0.232
$\mathbf{b}_{15}$	16.883	16.883	2036	3009	-0.004	0.033	0.033	0.946	0.244
$\mathbf{b}_{16}$	15.616	15.616	1700	2794	-0.002	0.035	0.035	0.940	0.258
$\mathbf{b}_{17}$	15.047	15.047	1543	2651	-0.002	0.035	0.035	0.944	0.263
$\mathbf{b}_{18}$	15.155	15.155	1525	2644	-0.001	0.034	0.034	0.941	0.260
$\mathbf{b}_{19}$	15.676	15.676	1553	2737	-0.001	0.033	0.033	0.944	0.254
$\mathbf{b}_{20}$	16.282	16.282	1566	2825	-0.001	0.033	0.033	0.946	0.250
$\mathbf{b}_{21}$	16.774	16.774	1539	2833	-0.001	0.033	0.033	0.947	0.250
`Uniform`								0.927	0.376
`WBATE`					-0.001	0.016	0.016	0.947	0.090
`LBATE`					0.020	0.027	0.033	0.987	0.387

Table SA-92: First-Stage Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.687	17.687	1868	3552	-0.004	0.030	0.030	0.950	0.230
$\mathbf{b}_2$	17.077	17.077	1823	3350	-0.004	0.031	0.031	0.947	0.236
$\mathbf{b}_3$	16.696	16.696	1810	3195	-0.003	0.031	0.031	0.950	0.241
$\mathbf{b}_4$	16.579	16.579	1851	3081	-0.002	0.032	0.032	0.946	0.243
$\mathbf{b}_5$	16.641	16.641	1943	2966	0.001	0.032	0.032	0.943	0.243
$\mathbf{b}_6$	16.776	16.776	2057	2824	0.003	0.032	0.032	0.944	0.244
$\mathbf{b}_7$	16.861	16.861	2154	2632	0.006	0.032	0.033	0.949	0.247
$\mathbf{b}_8$	16.634	16.634	2170	2336	0.007	0.034	0.034	0.944	0.256
$\mathbf{b}_9$	16.538	16.538	2193	2064	0.006	0.036	0.037	0.938	0.266
$\mathbf{b}_{10}$	16.270	16.270	2159	1750	0.002	0.041	0.041	0.944	0.292
$\mathbf{b}_{11}$	16.701	16.701	2235	1582	-0.003	0.047	0.047	0.945	0.360
$\mathbf{b}_{12}$	16.495	16.495	2199	1901	-0.006	0.038	0.039	0.945	0.275
$\mathbf{b}_{13}$	16.770	16.770	2209	2319	-0.006	0.033	0.034	0.947	0.253
$\mathbf{b}_{14}$	17.120	17.120	2193	2745	-0.005	0.031	0.032	0.950	0.238
$\mathbf{b}_{15}$	16.936	16.936	2037	2983	-0.004	0.031	0.031	0.945	0.235
$\mathbf{b}_{16}$	16.613	16.613	1856	3091	-0.003	0.031	0.031	0.947	0.236
$\mathbf{b}_{17}$	16.555	16.555	1767	3158	-0.002	0.031	0.031	0.952	0.236
$\mathbf{b}_{18}$	16.909	16.909	1769	3267	-0.002	0.030	0.030	0.949	0.232
$\mathbf{b}_{19}$	17.755	17.755	1827	3495	-0.002	0.029	0.029	0.949	0.224
$\mathbf{b}_{20}$	18.817	18.817	1881	3748	-0.002	0.029	0.029	0.940	0.216
$\mathbf{b}_{21}$	20.034	20.034	1917	3997	-0.001	0.027	0.027	0.945	0.209
`Uniform`								0.933	0.374
`WBATE`					-0.001	0.016	0.016	0.949	0.090
`LBATE`					0.015	0.024	0.029	0.990	0.375

Table SA-93: First-Stage Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.460	17.460	1835	3461	-0.004	0.030	0.031	0.951	0.232
$\mathbf{b}_2$	16.507	16.507	1732	3136	-0.004	0.031	0.032	0.948	0.242
$\mathbf{b}_3$	15.414	15.414	1590	2748	-0.003	0.033	0.033	0.952	0.257
$\mathbf{b}_4$	13.916	13.916	1371	2234	-0.002	0.037	0.037	0.946	0.284
$\mathbf{b}_5$	11.999	11.999	1082	1642	-0.000	0.043	0.043	0.943	0.331
$\mathbf{b}_6$	10.000	10.000	788	1113	0.001	0.053	0.053	0.939	0.400
$\mathbf{b}_7$	8.061	8.061	529	697	0.001	0.067	0.067	0.934	0.501
$\mathbf{b}_8$	7.300	7.300	450	532	0.002	0.078	0.078	0.921	0.569
$\mathbf{b}_9$	7.245	7.245	476	447	0.006	0.082	0.082	0.922	0.593
$\mathbf{b}_{10}$	7.128	7.128	500	343	0.005	0.088	0.088	0.922	0.633
$\mathbf{b}_{11}$	7.317	7.317	563	261	-0.001	0.112	0.112	0.902	0.832
$\mathbf{b}_{12}$	7.227	7.227	497	390	-0.002	0.083	0.083	0.921	0.609
$\mathbf{b}_{13}$	7.348	7.348	461	524	0.000	0.075	0.075	0.927	0.562
$\mathbf{b}_{14}$	8.409	8.409	573	765	-0.000	0.062	0.062	0.932	0.472
$\mathbf{b}_{15}$	11.200	11.200	962	1418	-0.001	0.046	0.046	0.936	0.348
$\mathbf{b}_{16}$	13.919	13.919	1371	2234	-0.001	0.036	0.036	0.943	0.277
$\mathbf{b}_{17}$	15.793	15.793	1640	2883	-0.002	0.032	0.032	0.951	0.245
$\mathbf{b}_{18}$	16.766	16.766	1748	3211	-0.002	0.030	0.030	0.949	0.234
$\mathbf{b}_{19}$	17.739	17.739	1825	3488	-0.002	0.029	0.029	0.949	0.225
$\mathbf{b}_{20}$	18.816	18.816	1881	3747	-0.002	0.029	0.029	0.940	0.216
$\mathbf{b}_{21}$	20.034	20.034	1917	3997	-0.001	0.027	0.027	0.945	0.209
`Uniform`								0.862	0.603
`WBATE`					-0.000	0.020	0.020	0.941	0.115
`LBATE`					0.058	0.049	0.076	0.959	0.769

Table SA-94: First-Stage Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.749	7.749	513	675	-0.002	0.071	0.071	0.941	0.272
$\mathbf{b}_2$	7.482	7.482	490	638	-0.002	0.072	0.072	0.946	0.280
$\mathbf{b}_3$	7.315	7.315	474	614	-0.001	0.072	0.072	0.950	0.285
$\mathbf{b}_4$	7.264	7.264	467	604	-0.000	0.074	0.074	0.944	0.286
$\mathbf{b}_5$	7.290	7.290	465	602	0.002	0.074	0.074	0.944	0.286
$\mathbf{b}_6$	7.350	7.350	463	600	0.002	0.073	0.073	0.944	0.285
$\mathbf{b}_7$	7.387	7.387	455	589	0.001	0.074	0.074	0.944	0.286
$\mathbf{b}_8$	7.287	7.287	449	530	0.002	0.078	0.078	0.932	0.296
$\mathbf{b}_9$	7.245	7.245	476	447	0.006	0.082	0.082	0.937	0.309
$\mathbf{b}_{10}$	7.128	7.128	500	343	0.005	0.088	0.088	0.934	0.337
$\mathbf{b}_{11}$	7.317	7.317	563	261	-0.001	0.112	0.112	0.938	0.432
$\mathbf{b}_{12}$	7.227	7.227	497	390	-0.002	0.083	0.083	0.941	0.318
$\mathbf{b}_{13}$	7.347	7.347	461	523	0.000	0.075	0.075	0.942	0.292
$\mathbf{b}_{14}$	7.501	7.501	469	610	0.000	0.071	0.071	0.947	0.276
$\mathbf{b}_{15}$	7.420	7.420	477	619	0.001	0.072	0.072	0.938	0.275
$\mathbf{b}_{16}$	7.278	7.278	468	606	0.002	0.072	0.072	0.938	0.278
$\mathbf{b}_{17}$	7.253	7.253	465	601	-0.001	0.072	0.072	0.948	0.279
$\mathbf{b}_{18}$	7.408	7.408	475	617	-0.002	0.070	0.070	0.947	0.275
$\mathbf{b}_{19}$	7.779	7.779	502	661	-0.001	0.068	0.068	0.945	0.265
$\mathbf{b}_{20}$	8.244	8.244	530	710	-0.001	0.067	0.067	0.943	0.255
$\mathbf{b}_{21}$	8.777	8.777	556	759	-0.001	0.064	0.064	0.945	0.246
`Uniform`								0.905	0.445
`WBATE`					0.000	0.024	0.024	0.948	0.092
`LBATE`					0.091	0.048	0.103	0.977	0.444

Table SA-95: First-Stage Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	16.600	16.600	1719	3154	-0.004	0.032	0.033	0.950	0.246
$\mathbf{b}_2$	16.166	16.166	1690	3035	-0.003	0.033	0.033	0.946	0.252
$\mathbf{b}_3$	15.882	15.882	1686	2931	-0.003	0.034	0.034	0.953	0.257
$\mathbf{b}_4$	15.852	15.852	1738	2864	-0.001	0.035	0.035	0.949	0.260
$\mathbf{b}_5$	16.025	16.025	1840	2807	0.000	0.035	0.035	0.944	0.260
$\mathbf{b}_6$	16.350	16.350	1980	2738	0.003	0.035	0.035	0.945	0.258
$\mathbf{b}_7$	16.274	16.274	2037	2515	0.006	0.035	0.036	0.949	0.263
$\mathbf{b}_8$	16.244	16.244	2091	2279	0.007	0.037	0.037	0.943	0.270
$\mathbf{b}_9$	16.123	16.123	2107	2009	0.006	0.038	0.039	0.940	0.280
$\mathbf{b}_{10}$	15.915	15.915	2087	1714	0.002	0.042	0.042	0.940	0.305
$\mathbf{b}_{11}$	16.264	16.264	2148	1536	-0.003	0.050	0.050	0.942	0.380
$\mathbf{b}_{12}$	16.237	16.237	2145	1881	-0.006	0.040	0.040	0.944	0.286
$\mathbf{b}_{13}$	16.776	16.776	2209	2356	-0.006	0.035	0.035	0.944	0.258
$\mathbf{b}_{14}$	16.977	16.977	2166	2747	-0.006	0.033	0.033	0.946	0.248
$\mathbf{b}_{15}$	16.647	16.647	1996	2940	-0.004	0.033	0.034	0.944	0.249
$\mathbf{b}_{16}$	15.901	15.901	1757	2891	-0.002	0.034	0.034	0.942	0.256
$\mathbf{b}_{17}$	15.588	15.588	1630	2851	-0.002	0.034	0.034	0.950	0.257
$\mathbf{b}_{18}$	15.786	15.786	1612	2884	-0.002	0.033	0.033	0.947	0.252
$\mathbf{b}_{19}$	16.349	16.349	1641	2994	-0.002	0.032	0.032	0.949	0.246
$\mathbf{b}_{20}$	17.026	17.026	1658	3109	-0.002	0.032	0.032	0.941	0.241
$\mathbf{b}_{21}$	17.757	17.757	1651	3197	-0.001	0.032	0.032	0.945	0.239
`Uniform`								0.927	0.401
`WBATE`					-0.001	0.016	0.016	0.947	0.093
`LBATE`					0.021	0.027	0.034	0.987	0.420

Table SA-96: First-Stage Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.940	18.940	2038	4030	0.002	0.056	0.056	0.958	0.408
$\mathbf{b}_2$	18.560	18.560	2046	3894	0.002	0.057	0.057	0.960	0.412
$\mathbf{b}_3$	18.341	18.341	2087	3764	0.003	0.058	0.058	0.959	0.414
$\mathbf{b}_4$	18.247	18.247	2164	3620	0.005	0.060	0.060	0.960	0.415
$\mathbf{b}_5$	18.265	18.265	2269	3460	0.009	0.060	0.060	0.962	0.414
$\mathbf{b}_6$	18.404	18.404	2394	3295	0.013	0.059	0.060	0.959	0.414
$\mathbf{b}_7$	18.492	18.492	2494	3084	0.016	0.059	0.061	0.958	0.418
$\mathbf{b}_8$	18.409	18.409	2536	2801	0.015	0.060	0.062	0.956	0.427
$\mathbf{b}_9$	18.445	18.445	2577	2534	0.012	0.061	0.062	0.958	0.435
$\mathbf{b}_{10}$	18.570	18.570	2606	2280	0.007	0.062	0.063	0.967	0.447
$\mathbf{b}_{11}$	20.183	20.183	2867	2376	0.002	0.063	0.063	0.968	0.461
$\mathbf{b}_{12}$	18.471	18.471	2589	2371	-0.005	0.061	0.062	0.962	0.437
$\mathbf{b}_{13}$	18.461	18.461	2555	2759	-0.012	0.060	0.061	0.958	0.415
$\mathbf{b}_{14}$	18.567	18.567	2494	3154	-0.013	0.057	0.059	0.956	0.394
$\mathbf{b}_{15}$	18.485	18.485	2354	3449	-0.009	0.056	0.057	0.956	0.382
$\mathbf{b}_{16}$	18.323	18.323	2178	3645	-0.003	0.055	0.055	0.962	0.380
$\mathbf{b}_{17}$	18.304	18.304	2048	3779	0.001	0.052	0.052	0.963	0.378
$\mathbf{b}_{18}$	18.620	18.620	2006	3912	0.002	0.050	0.050	0.959	0.372
$\mathbf{b}_{19}$	19.300	19.300	2021	4091	0.001	0.048	0.048	0.952	0.361
$\mathbf{b}_{20}$	20.308	20.308	2057	4323	-0.001	0.048	0.048	0.955	0.347
$\mathbf{b}_{21}$	21.402	21.402	2067	4519	-0.003	0.047	0.047	0.954	0.336
`Uniform`								0.971	0.606
`WBATE`					0.002	0.027	0.027	0.954	0.148
`LBATE`					0.055	0.045	0.071	0.996	0.670

Table SA-97: Fuzzy Effects (Simulations, Linear Homoskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.446	18.446	1967	3834	0.003	0.057	0.057	0.958	0.417
$\mathbf{b}_2$	17.301	17.301	1843	3422	0.003	0.059	0.059	0.957	0.438
$\mathbf{b}_3$	15.833	15.833	1650	2887	0.003	0.063	0.063	0.954	0.474
$\mathbf{b}_4$	13.993	13.993	1382	2255	0.003	0.070	0.070	0.958	0.535
$\mathbf{b}_5$	12.000	12.000	1081	1641	0.001	0.081	0.081	0.961	0.629
$\mathbf{b}_6$	10.000	10.000	788	1113	-0.000	0.097	0.097	0.957	0.774
$\mathbf{b}_7$	8.375	8.375	570	751	0.001	0.120	0.120	0.967	0.985
$\mathbf{b}_8$	8.066	8.066	554	636	0.004	0.128	0.128	0.967	1.097
$\mathbf{b}_9$	8.081	8.081	595	543	0.008	0.129	0.129	0.976	1.468
$\mathbf{b}_{10}$	8.136	8.136	646	441	0.008	0.133	0.133	0.972	1.622
$\mathbf{b}_{11}$	8.842	8.842	794	394	0.003	0.143	0.143	0.978	28.691
$\mathbf{b}_{12}$	8.092	8.092	623	480	-0.005	0.131	0.131	0.972	1.302
$\mathbf{b}_{13}$	8.088	8.088	564	622	-0.002	0.126	0.126	0.970	1.010
$\mathbf{b}_{14}$	8.563	8.563	593	793	0.004	0.111	0.111	0.966	0.884
$\mathbf{b}_{15}$	11.200	11.200	961	1419	0.004	0.083	0.083	0.958	0.631
$\mathbf{b}_{16}$	13.993	13.993	1381	2257	0.002	0.064	0.064	0.954	0.491
$\mathbf{b}_{17}$	16.467	16.467	1737	3117	0.001	0.055	0.055	0.958	0.414
$\mathbf{b}_{18}$	18.090	18.090	1927	3704	0.001	0.051	0.051	0.959	0.381
$\mathbf{b}_{19}$	19.259	19.259	2016	4074	0.001	0.048	0.048	0.952	0.361
$\mathbf{b}_{20}$	20.308	20.308	2057	4323	-0.001	0.048	0.048	0.955	0.347
$\mathbf{b}_{21}$	21.402	21.402	2067	4519	-0.003	0.047	0.047	0.954	0.336
`Uniform`								0.978	3.166
`WBATE`					0.002	0.034	0.034	0.960	1.575
`LBATE`					0.139	0.090	0.166	0.997	23.429

Table SA-98: Fuzzy Effects (Simulations, Linear Homoskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.298	8.298	578	774	0.005	0.121	0.121	0.955	0.471
$\mathbf{b}_2$	8.131	8.131	566	754	0.005	0.123	0.123	0.952	0.476
$\mathbf{b}_3$	8.035	8.035	559	741	0.004	0.123	0.123	0.957	0.480
$\mathbf{b}_4$	7.994	7.994	553	733	0.002	0.125	0.125	0.959	0.481
$\mathbf{b}_5$	8.002	8.002	549	726	0.000	0.123	0.123	0.959	0.481
$\mathbf{b}_6$	8.063	8.063	546	723	-0.002	0.124	0.124	0.955	0.481
$\mathbf{b}_7$	8.101	8.101	540	706	0.000	0.125	0.125	0.957	0.486
$\mathbf{b}_8$	8.065	8.065	554	636	0.004	0.128	0.128	0.959	0.495
$\mathbf{b}_9$	8.081	8.081	595	543	0.008	0.129	0.129	0.963	0.506
$\mathbf{b}_{10}$	8.136	8.136	646	441	0.008	0.133	0.133	0.961	0.521
$\mathbf{b}_{11}$	8.842	8.842	794	394	0.003	0.143	0.143	0.963	0.551
$\mathbf{b}_{12}$	8.092	8.092	623	480	-0.005	0.131	0.131	0.960	0.509
$\mathbf{b}_{13}$	8.088	8.088	564	622	-0.002	0.126	0.126	0.959	0.481
$\mathbf{b}_{14}$	8.134	8.134	544	719	0.004	0.118	0.118	0.958	0.458
$\mathbf{b}_{15}$	8.098	8.098	556	739	0.004	0.117	0.117	0.957	0.445
$\mathbf{b}_{16}$	8.028	8.028	556	738	0.001	0.114	0.114	0.954	0.440
$\mathbf{b}_{17}$	8.019	8.019	555	737	-0.001	0.111	0.111	0.958	0.435
$\mathbf{b}_{18}$	8.157	8.157	562	751	0.000	0.109	0.109	0.955	0.429
$\mathbf{b}_{19}$	8.456	8.456	580	782	0.002	0.108	0.108	0.952	0.414
$\mathbf{b}_{20}$	8.897	8.897	604	827	0.000	0.102	0.102	0.954	0.397
$\mathbf{b}_{21}$	9.376	9.376	621	865	-0.002	0.099	0.099	0.953	0.383
`Uniform`								0.973	0.713
`WBATE`					0.002	0.039	0.039	0.954	0.153
`LBATE`					0.184	0.089	0.204	0.997	0.815

Table SA-99: Fuzzy Effects (Simulations, Linear Homoskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.446	18.446	1976	3846	0.000	0.060	0.060	0.960	0.423
$\mathbf{b}_2$	18.337	18.337	2020	3819	0.000	0.061	0.061	0.958	0.422
$\mathbf{b}_3$	18.348	18.348	2101	3775	0.001	0.062	0.062	0.962	0.420
$\mathbf{b}_4$	18.488	18.488	2219	3716	0.003	0.063	0.063	0.962	0.418
$\mathbf{b}_5$	18.737	18.737	2365	3637	0.007	0.064	0.064	0.966	0.414
$\mathbf{b}_6$	18.966	18.966	2501	3501	0.010	0.064	0.064	0.958	0.413
$\mathbf{b}_7$	19.098	19.098	2604	3300	0.012	0.064	0.065	0.959	0.417
$\mathbf{b}_8$	19.288	19.288	2695	3089	0.012	0.065	0.066	0.959	0.419
$\mathbf{b}_9$	19.613	19.613	2782	2892	0.010	0.064	0.065	0.964	0.420
$\mathbf{b}_{10}$	20.205	20.205	2884	2747	0.006	0.063	0.064	0.966	0.423
$\mathbf{b}_{11}$	21.806	21.806	3110	2852	0.001	0.065	0.065	0.967	0.439
$\mathbf{b}_{12}$	19.941	19.941	2845	2797	-0.005	0.064	0.064	0.967	0.418
$\mathbf{b}_{13}$	19.520	19.520	2749	3093	-0.011	0.063	0.064	0.960	0.405
$\mathbf{b}_{14}$	19.298	19.298	2631	3407	-0.011	0.061	0.062	0.958	0.389
$\mathbf{b}_{15}$	19.009	19.009	2456	3648	-0.006	0.062	0.062	0.962	0.384
$\mathbf{b}_{16}$	18.515	18.515	2227	3730	-0.000	0.060	0.060	0.962	0.387
$\mathbf{b}_{17}$	18.115	18.115	2033	3722	0.004	0.056	0.056	0.969	0.390
$\mathbf{b}_{18}$	18.047	18.047	1931	3705	0.004	0.054	0.054	0.963	0.389
$\mathbf{b}_{19}$	18.280	18.280	1892	3705	0.003	0.053	0.053	0.954	0.383
$\mathbf{b}_{20}$	18.825	18.825	1878	3762	0.002	0.054	0.054	0.957	0.376
$\mathbf{b}_{21}$	19.374	19.374	1839	3766	-0.000	0.054	0.054	0.956	0.372
`Uniform`								0.976	0.610
`WBATE`					0.002	0.027	0.028	0.949	0.149
`LBATE`					0.060	0.051	0.079	0.998	0.707

Table SA-100: Fuzzy Effects (Simulations, Linear Homoskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	19.488	19.488	2114	4244	-0.000	0.074	0.074	0.963	0.530
$\mathbf{b}_2$	18.846	18.846	2089	4002	-0.000	0.075	0.075	0.960	0.534
$\mathbf{b}_3$	18.345	18.345	2087	3766	0.001	0.076	0.076	0.963	0.537
$\mathbf{b}_4$	17.972	17.972	2110	3530	0.004	0.076	0.076	0.960	0.539
$\mathbf{b}_5$	17.684	17.684	2151	3279	0.008	0.077	0.077	0.957	0.541
$\mathbf{b}_6$	17.475	17.475	2202	3021	0.012	0.077	0.078	0.964	0.543
$\mathbf{b}_7$	17.293	17.293	2244	2747	0.014	0.078	0.080	0.960	0.548
$\mathbf{b}_8$	16.916	16.916	2229	2406	0.014	0.080	0.081	0.959	0.562
$\mathbf{b}_9$	16.732	16.732	2233	2109	0.012	0.080	0.081	0.958	0.571
$\mathbf{b}_{10}$	16.448	16.448	2194	1787	0.006	0.078	0.078	0.958	0.571
$\mathbf{b}_{11}$	16.857	16.857	2265	1614	0.001	0.074	0.074	0.964	0.534
$\mathbf{b}_{12}$	16.613	16.613	2223	1927	-0.005	0.080	0.080	0.961	0.587
$\mathbf{b}_{13}$	16.893	16.893	2234	2351	-0.012	0.084	0.085	0.957	0.584
$\mathbf{b}_{14}$	17.334	17.334	2239	2803	-0.014	0.085	0.086	0.956	0.571
$\mathbf{b}_{15}$	17.565	17.565	2165	3168	-0.009	0.085	0.086	0.953	0.573
$\mathbf{b}_{16}$	17.929	17.929	2101	3515	-0.003	0.084	0.084	0.955	0.578
$\mathbf{b}_{17}$	18.505	18.505	2078	3852	0.000	0.083	0.083	0.955	0.582
$\mathbf{b}_{18}$	19.329	19.329	2103	4188	0.001	0.084	0.084	0.955	0.584
$\mathbf{b}_{19}$	20.492	20.492	2167	4569	0.000	0.083	0.083	0.954	0.581
$\mathbf{b}_{20}$	21.968	21.968	2239	4998	-0.001	0.083	0.083	0.958	0.575
$\mathbf{b}_{21}$	23.598	23.598	2285	5410	-0.003	0.083	0.083	0.958	0.572
`Uniform`								0.971	0.847
`WBATE`					0.001	0.034	0.034	0.949	0.193
`LBATE`					0.085	0.060	0.104	0.996	0.934

Table SA-101: Fuzzy Effects (Simulations, Linear Heteroskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.811	18.811	2015	3976	0.001	0.075	0.075	0.962	0.545
$\mathbf{b}_2$	17.441	17.441	1863	3475	0.002	0.078	0.078	0.958	0.572
$\mathbf{b}_3$	15.849	15.849	1652	2894	0.002	0.082	0.082	0.960	0.615
$\mathbf{b}_4$	13.990	13.990	1381	2255	0.002	0.090	0.090	0.954	0.687
$\mathbf{b}_5$	12.000	12.000	1081	1641	0.000	0.103	0.103	0.957	0.798
$\mathbf{b}_6$	10.000	10.000	788	1112	-0.002	0.123	0.123	0.961	0.966
$\mathbf{b}_7$	8.099	8.099	534	703	-0.003	0.154	0.154	0.966	1.279
$\mathbf{b}_8$	7.421	7.421	465	547	0.002	0.169	0.169	0.973	1.476
$\mathbf{b}_9$	7.330	7.330	487	456	0.008	0.179	0.179	0.972	1.959
$\mathbf{b}_{10}$	7.206	7.206	510	349	0.009	0.178	0.178	0.973	6.927
$\mathbf{b}_{11}$	7.385	7.385	572	266	0.004	0.168	0.168	0.984	8383.074
$\mathbf{b}_{12}$	7.278	7.278	504	395	-0.005	0.179	0.179	0.975	37.023
$\mathbf{b}_{13}$	7.402	7.402	469	531	-0.002	0.178	0.178	0.972	1.650
$\mathbf{b}_{14}$	8.414	8.414	574	766	-0.000	0.155	0.155	0.959	4.888
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.002	0.120	0.120	0.950	0.897
$\mathbf{b}_{16}$	13.989	13.989	1381	2256	-0.001	0.098	0.098	0.952	0.733
$\mathbf{b}_{17}$	16.557	16.557	1750	3151	-0.001	0.087	0.087	0.950	0.644
$\mathbf{b}_{18}$	18.615	18.615	1998	3908	-0.000	0.085	0.085	0.952	0.603
$\mathbf{b}_{19}$	20.332	20.332	2147	4503	-0.000	0.084	0.084	0.953	0.584
$\mathbf{b}_{20}$	21.947	21.947	2236	4989	-0.001	0.083	0.083	0.958	0.575
$\mathbf{b}_{21}$	23.593	23.593	2284	5408	-0.003	0.083	0.083	0.958	0.572
`Uniform`								0.978	625.773
`WBATE`					0.000	0.044	0.044	0.961	401.556
`LBATE`					0.201	0.125	0.237	0.998	6336.887

Table SA-102: Fuzzy Effects (Simulations, Linear Heteroskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	8.538	8.538	607	820	0.001	0.155	0.155	0.958	0.603
$\mathbf{b}_2$	8.257	8.257	582	778	0.002	0.156	0.156	0.958	0.610
$\mathbf{b}_3$	8.037	8.037	559	741	0.003	0.159	0.159	0.957	0.617
$\mathbf{b}_4$	7.874	7.874	538	709	0.001	0.162	0.162	0.955	0.622
$\mathbf{b}_5$	7.747	7.747	518	679	-0.002	0.163	0.163	0.957	0.629
$\mathbf{b}_6$	7.656	7.656	498	650	-0.004	0.165	0.165	0.960	0.634
$\mathbf{b}_7$	7.576	7.576	477	618	-0.003	0.166	0.166	0.958	0.642
$\mathbf{b}_8$	7.411	7.411	465	546	0.002	0.170	0.170	0.960	0.656
$\mathbf{b}_9$	7.330	7.330	487	456	0.008	0.179	0.179	0.954	0.669
$\mathbf{b}_{10}$	7.206	7.206	510	349	0.009	0.178	0.178	0.957	0.673
$\mathbf{b}_{11}$	7.385	7.385	572	266	0.004	0.168	0.168	0.967	0.632
$\mathbf{b}_{12}$	7.278	7.278	504	395	-0.005	0.179	0.179	0.959	0.689
$\mathbf{b}_{13}$	7.401	7.401	468	530	-0.002	0.178	0.178	0.953	0.679
$\mathbf{b}_{14}$	7.594	7.594	480	625	-0.000	0.175	0.175	0.957	0.666
$\mathbf{b}_{15}$	7.695	7.695	508	665	-0.003	0.174	0.174	0.953	0.663
$\mathbf{b}_{16}$	7.855	7.855	536	705	-0.000	0.172	0.172	0.954	0.661
$\mathbf{b}_{17}$	8.107	8.107	566	752	-0.000	0.169	0.169	0.954	0.659
$\mathbf{b}_{18}$	8.468	8.468	601	809	-0.001	0.173	0.173	0.946	0.657
$\mathbf{b}_{19}$	8.978	8.978	644	883	-0.002	0.168	0.168	0.951	0.648
$\mathbf{b}_{20}$	9.624	9.624	692	970	0.000	0.163	0.163	0.955	0.638
$\mathbf{b}_{21}$	10.338	10.338	734	1056	0.001	0.165	0.165	0.955	0.632
`Uniform`								0.970	0.989
`WBATE`					0.000	0.053	0.053	0.947	0.202
`LBATE`					0.268	0.131	0.298	0.995	1.125

Table SA-103: Fuzzy Effects (Simulations, Linear Heteroskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	18.343	18.343	1960	3818	-0.002	0.086	0.086	0.963	0.569
$\mathbf{b}_2$	18.051	18.051	1977	3719	-0.002	0.085	0.085	0.963	0.566
$\mathbf{b}_3$	17.855	17.855	2018	3608	-0.002	0.086	0.086	0.965	0.564
$\mathbf{b}_4$	17.785	17.785	2090	3485	0.001	0.086	0.086	0.960	0.558
$\mathbf{b}_5$	17.710	17.710	2166	3313	0.004	0.087	0.087	0.962	0.555
$\mathbf{b}_6$	17.660	17.660	2243	3107	0.008	0.088	0.088	0.967	0.554
$\mathbf{b}_7$	17.157	17.157	2217	2745	0.010	0.090	0.091	0.965	0.568
$\mathbf{b}_8$	16.803	16.803	2205	2415	0.011	0.090	0.091	0.967	0.580
$\mathbf{b}_9$	16.462	16.462	2176	2085	0.009	0.092	0.093	0.962	0.600
$\mathbf{b}_{10}$	16.219	16.219	2146	1778	0.005	0.090	0.090	0.966	0.599
$\mathbf{b}_{11}$	16.492	16.492	2191	1581	0.001	0.085	0.085	0.970	0.594
$\mathbf{b}_{12}$	16.520	16.520	2200	1947	-0.005	0.092	0.092	0.970	0.607
$\mathbf{b}_{13}$	17.243	17.243	2302	2481	-0.010	0.094	0.095	0.963	0.591
$\mathbf{b}_{14}$	17.877	17.877	2349	2999	-0.010	0.095	0.095	0.962	0.572
$\mathbf{b}_{15}$	18.466	18.466	2354	3483	-0.005	0.096	0.096	0.960	0.567
$\mathbf{b}_{16}$	18.673	18.673	2263	3788	-0.000	0.095	0.095	0.963	0.577
$\mathbf{b}_{17}$	18.855	18.855	2164	3992	0.003	0.094	0.094	0.963	0.589
$\mathbf{b}_{18}$	19.160	19.160	2095	4145	0.004	0.098	0.098	0.966	0.604
$\mathbf{b}_{19}$	19.705	19.705	2067	4290	0.004	0.100	0.100	0.959	0.615
$\mathbf{b}_{20}$	20.543	20.543	2067	4467	0.004	0.098	0.099	0.962	0.623
$\mathbf{b}_{21}$	21.462	21.462	2050	4606	0.003	0.102	0.102	0.965	0.637
`Uniform`								0.978	0.884
`WBATE`					0.001	0.038	0.038	0.944	0.200
`LBATE`					0.109	0.083	0.137	0.998	1.106

Table SA-104: Fuzzy Effects (Simulations, Linear Heteroskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.240	17.240	1807	3381	0.001	0.062	0.062	0.954	0.458
$\mathbf{b}_2$	16.756	16.756	1775	3231	0.000	0.063	0.063	0.954	0.464
$\mathbf{b}_3$	16.586	16.586	1790	3156	0.001	0.064	0.064	0.961	0.462
$\mathbf{b}_4$	16.714	16.714	1873	3125	0.004	0.064	0.065	0.960	0.455
$\mathbf{b}_5$	17.111	17.111	2036	3110	0.012	0.065	0.066	0.957	0.441
$\mathbf{b}_6$	17.598	17.598	2226	3064	0.022	0.062	0.066	0.952	0.425
$\mathbf{b}_7$	18.024	18.024	2395	2954	0.029	0.059	0.066	0.958	0.412
$\mathbf{b}_8$	18.133	18.133	2479	2728	0.029	0.057	0.064	0.954	0.407
$\mathbf{b}_9$	18.233	18.233	2535	2481	0.023	0.056	0.061	0.960	0.405
$\mathbf{b}_{10}$	18.423	18.423	2578	2245	0.014	0.057	0.058	0.960	0.410
$\mathbf{b}_{11}$	20.110	20.110	2855	2358	0.004	0.056	0.056	0.958	0.414
$\mathbf{b}_{12}$	18.393	18.393	2574	2351	-0.007	0.056	0.056	0.957	0.395
$\mathbf{b}_{13}$	18.366	18.366	2537	2735	-0.019	0.054	0.057	0.953	0.377
$\mathbf{b}_{14}$	18.343	18.343	2448	3092	-0.023	0.053	0.058	0.953	0.364
$\mathbf{b}_{15}$	17.668	17.668	2187	3205	-0.016	0.054	0.057	0.959	0.371
$\mathbf{b}_{16}$	16.748	16.748	1880	3135	-0.004	0.055	0.055	0.959	0.387
$\mathbf{b}_{17}$	16.162	16.162	1707	3017	-0.001	0.053	0.053	0.956	0.399
$\mathbf{b}_{18}$	16.176	16.176	1668	2995	-0.001	0.053	0.053	0.956	0.397
$\mathbf{b}_{19}$	16.736	16.736	1694	3114	-0.002	0.052	0.053	0.952	0.388
$\mathbf{b}_{20}$	17.551	17.551	1727	3273	-0.004	0.052	0.052	0.952	0.377
$\mathbf{b}_{21}$	18.491	18.491	1744	3422	-0.007	0.051	0.051	0.955	0.368
`Uniform`								0.966	0.616
`WBATE`					0.003	0.026	0.026	0.951	0.146
`LBATE`					0.064	0.043	0.077	0.994	0.661

Table SA-105: Fuzzy Effects (Simulations, Quadratic Homoskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.072	17.072	1783	3314	0.002	0.062	0.062	0.955	0.461
$\mathbf{b}_2$	16.314	16.314	1705	3064	0.001	0.064	0.064	0.954	0.473
$\mathbf{b}_3$	15.415	15.415	1589	2746	0.002	0.066	0.066	0.959	0.491
$\mathbf{b}_4$	13.947	13.947	1374	2243	0.002	0.071	0.071	0.955	0.537
$\mathbf{b}_5$	12.000	12.000	1080	1642	0.002	0.081	0.081	0.958	0.622
$\mathbf{b}_6$	10.000	10.000	787	1114	0.001	0.096	0.096	0.962	0.752
$\mathbf{b}_7$	8.292	8.292	559	737	0.000	0.117	0.117	0.966	0.937
$\mathbf{b}_8$	7.946	7.946	537	620	0.004	0.123	0.123	0.970	1.013
$\mathbf{b}_9$	7.988	7.988	582	533	0.012	0.123	0.123	0.964	1.032
$\mathbf{b}_{10}$	8.071	8.071	636	435	0.013	0.123	0.124	0.971	1.018
$\mathbf{b}_{11}$	8.811	8.811	788	391	0.005	0.130	0.130	0.968	11.321
$\mathbf{b}_{12}$	8.058	8.058	618	477	-0.004	0.119	0.119	0.965	0.958
$\mathbf{b}_{13}$	8.047	8.047	558	616	-0.001	0.113	0.113	0.969	2.225
$\mathbf{b}_{14}$	8.537	8.537	590	788	0.001	0.103	0.103	0.963	0.797
$\mathbf{b}_{15}$	11.200	11.200	962	1419	-0.001	0.076	0.076	0.955	0.581
$\mathbf{b}_{16}$	13.954	13.954	1376	2245	-0.002	0.060	0.060	0.953	0.458
$\mathbf{b}_{17}$	15.604	15.604	1614	2816	-0.002	0.054	0.054	0.955	0.410
$\mathbf{b}_{18}$	16.095	16.095	1656	2963	-0.002	0.053	0.053	0.957	0.399
$\mathbf{b}_{19}$	16.730	16.730	1693	3111	-0.002	0.053	0.053	0.952	0.388
$\mathbf{b}_{20}$	17.551	17.551	1727	3273	-0.004	0.052	0.052	0.952	0.377
$\mathbf{b}_{21}$	18.491	18.491	1744	3422	-0.007	0.051	0.051	0.955	0.368
`Uniform`								0.976	1.868
`WBATE`					0.001	0.032	0.032	0.953	0.752
`LBATE`					0.137	0.079	0.158	0.997	10.423

Table SA-106: Fuzzy Effects (Simulations, Quadratic Homoskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.553	7.553	489	640	0.001	0.137	0.137	0.951	0.530
$\mathbf{b}_2$	7.341	7.341	472	613	0.001	0.142	0.142	0.952	0.539
$\mathbf{b}_3$	7.267	7.267	467	605	0.000	0.138	0.138	0.961	0.538
$\mathbf{b}_4$	7.322	7.322	473	615	-0.001	0.138	0.138	0.955	0.530
$\mathbf{b}_5$	7.496	7.496	489	639	0.000	0.132	0.132	0.958	0.511
$\mathbf{b}_6$	7.710	7.710	505	664	0.001	0.128	0.128	0.955	0.493
$\mathbf{b}_7$	7.896	7.896	516	673	-0.000	0.125	0.125	0.957	0.480
$\mathbf{b}_8$	7.944	7.944	537	620	0.004	0.123	0.123	0.955	0.476
$\mathbf{b}_9$	7.988	7.988	582	533	0.012	0.123	0.123	0.954	0.474
$\mathbf{b}_{10}$	8.071	8.071	636	435	0.013	0.123	0.124	0.955	0.477
$\mathbf{b}_{11}$	8.811	8.811	788	391	0.005	0.130	0.130	0.954	0.495
$\mathbf{b}_{12}$	8.058	8.058	618	477	-0.004	0.119	0.119	0.952	0.460
$\mathbf{b}_{13}$	8.046	8.046	558	616	-0.001	0.113	0.113	0.956	0.437
$\mathbf{b}_{14}$	8.036	8.036	533	702	0.002	0.111	0.111	0.952	0.424
$\mathbf{b}_{15}$	7.741	7.741	515	675	-0.002	0.111	0.111	0.954	0.431
$\mathbf{b}_{16}$	7.337	7.337	476	616	-0.002	0.117	0.117	0.952	0.449
$\mathbf{b}_{17}$	7.081	7.081	445	572	-0.006	0.119	0.120	0.952	0.463
$\mathbf{b}_{18}$	7.087	7.087	438	562	-0.003	0.117	0.117	0.956	0.461
$\mathbf{b}_{19}$	7.332	7.332	451	585	0.001	0.117	0.117	0.951	0.449
$\mathbf{b}_{20}$	7.689	7.689	468	614	-0.001	0.114	0.114	0.950	0.436
$\mathbf{b}_{21}$	8.101	8.101	482	641	-0.003	0.108	0.108	0.957	0.425
`Uniform`								0.966	0.727
`WBATE`					0.001	0.039	0.039	0.947	0.151
`LBATE`					0.186	0.082	0.203	0.995	0.801

Table SA-107: Fuzzy Effects (Simulations, Quadratic Homoskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	16.410	16.410	1693	3079	-0.002	0.066	0.066	0.957	0.482
$\mathbf{b}_2$	16.047	16.047	1671	2986	-0.003	0.068	0.068	0.956	0.487
$\mathbf{b}_3$	15.948	15.948	1694	2949	-0.003	0.069	0.069	0.965	0.486
$\mathbf{b}_4$	16.089	16.089	1775	2937	-0.000	0.071	0.071	0.966	0.482
$\mathbf{b}_5$	16.495	16.495	1926	2946	0.006	0.073	0.073	0.965	0.468
$\mathbf{b}_6$	17.158	17.158	2138	2971	0.015	0.072	0.074	0.957	0.449
$\mathbf{b}_7$	17.721	17.721	2326	2913	0.022	0.068	0.072	0.959	0.433
$\mathbf{b}_8$	18.359	18.359	2512	2837	0.024	0.064	0.069	0.955	0.415
$\mathbf{b}_9$	18.990	18.990	2663	2730	0.020	0.060	0.063	0.960	0.401
$\mathbf{b}_{10}$	19.803	19.803	2812	2645	0.012	0.059	0.060	0.964	0.392
$\mathbf{b}_{11}$	21.462	21.462	3055	2761	0.003	0.060	0.060	0.963	0.400
$\mathbf{b}_{12}$	19.640	19.640	2792	2714	-0.007	0.058	0.058	0.960	0.379
$\mathbf{b}_{13}$	19.005	19.005	2650	2951	-0.017	0.058	0.060	0.956	0.373
$\mathbf{b}_{14}$	18.130	18.130	2396	3075	-0.018	0.060	0.063	0.959	0.380
$\mathbf{b}_{15}$	16.883	16.883	2036	3009	-0.009	0.064	0.064	0.962	0.402
$\mathbf{b}_{16}$	15.616	15.616	1700	2794	0.002	0.063	0.063	0.964	0.426
$\mathbf{b}_{17}$	15.047	15.047	1543	2651	0.003	0.059	0.060	0.960	0.433
$\mathbf{b}_{18}$	15.155	15.155	1525	2644	0.002	0.058	0.058	0.959	0.425
$\mathbf{b}_{19}$	15.676	15.676	1553	2737	0.002	0.058	0.058	0.956	0.413
$\mathbf{b}_{20}$	16.282	16.282	1566	2825	-0.000	0.059	0.059	0.957	0.406
$\mathbf{b}_{21}$	16.774	16.774	1539	2833	-0.003	0.059	0.059	0.960	0.406
`Uniform`								0.972	0.643
`WBATE`					0.002	0.028	0.028	0.949	0.149
`LBATE`					0.068	0.046	0.083	0.996	0.712

Table SA-108: Fuzzy Effects (Simulations, Quadratic Homoskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.687	17.687	1868	3552	-0.001	0.071	0.071	0.957	0.512
$\mathbf{b}_2$	17.077	17.077	1823	3350	-0.001	0.071	0.071	0.955	0.510
$\mathbf{b}_3$	16.696	16.696	1810	3195	0.000	0.070	0.070	0.958	0.504
$\mathbf{b}_4$	16.579	16.579	1851	3081	0.004	0.071	0.071	0.961	0.492
$\mathbf{b}_5$	16.641	16.641	1943	2966	0.011	0.070	0.071	0.964	0.477
$\mathbf{b}_6$	16.776	16.776	2057	2824	0.021	0.068	0.071	0.958	0.461
$\mathbf{b}_7$	16.861	16.861	2154	2632	0.027	0.066	0.071	0.955	0.447
$\mathbf{b}_8$	16.634	16.634	2170	2336	0.028	0.064	0.070	0.955	0.445
$\mathbf{b}_9$	16.538	16.538	2193	2064	0.022	0.062	0.066	0.955	0.444
$\mathbf{b}_{10}$	16.270	16.270	2159	1750	0.013	0.060	0.062	0.963	0.438
$\mathbf{b}_{11}$	16.701	16.701	2235	1582	0.003	0.057	0.057	0.962	0.406
$\mathbf{b}_{12}$	16.495	16.495	2199	1901	-0.009	0.062	0.062	0.962	0.447
$\mathbf{b}_{13}$	16.770	16.770	2209	2319	-0.019	0.063	0.066	0.961	0.442
$\mathbf{b}_{14}$	17.120	17.120	2193	2745	-0.021	0.064	0.067	0.960	0.434
$\mathbf{b}_{15}$	16.936	16.936	2037	2983	-0.012	0.067	0.068	0.955	0.446
$\mathbf{b}_{16}$	16.613	16.613	1856	3091	-0.000	0.067	0.067	0.962	0.462
$\mathbf{b}_{17}$	16.555	16.555	1767	3158	0.003	0.067	0.067	0.953	0.477
$\mathbf{b}_{18}$	16.909	16.909	1769	3267	0.001	0.066	0.066	0.954	0.479
$\mathbf{b}_{19}$	17.755	17.755	1827	3495	-0.002	0.066	0.066	0.952	0.472
$\mathbf{b}_{20}$	18.817	18.817	1881	3748	-0.005	0.067	0.067	0.945	0.466
$\mathbf{b}_{21}$	20.034	20.034	1917	3997	-0.008	0.066	0.067	0.950	0.460
`Uniform`								0.968	0.700
`WBATE`					0.003	0.028	0.028	0.953	0.156
`LBATE`					0.074	0.048	0.088	0.995	0.745

Table SA-109: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, Smooth-Boundary Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	17.460	17.460	1835	3461	-0.000	0.072	0.072	0.958	0.517
$\mathbf{b}_2$	16.507	16.507	1732	3136	0.000	0.072	0.072	0.954	0.524
$\mathbf{b}_3$	15.414	15.414	1590	2748	0.001	0.073	0.073	0.954	0.540
$\mathbf{b}_4$	13.916	13.916	1371	2234	0.002	0.078	0.078	0.956	0.578
$\mathbf{b}_5$	11.999	11.999	1082	1642	0.002	0.086	0.086	0.961	0.653
$\mathbf{b}_6$	10.000	10.000	788	1113	0.002	0.099	0.099	0.958	0.776
$\mathbf{b}_7$	8.061	8.061	529	697	0.001	0.123	0.123	0.962	0.985
$\mathbf{b}_8$	7.300	7.300	450	532	0.002	0.136	0.136	0.971	1.285
$\mathbf{b}_9$	7.245	7.245	476	447	0.010	0.136	0.137	0.972	1.288
$\mathbf{b}_{10}$	7.128	7.128	500	343	0.010	0.133	0.134	0.970	2.205
$\mathbf{b}_{11}$	7.317	7.317	563	261	0.002	0.124	0.124	0.981	10.738
$\mathbf{b}_{12}$	7.227	7.227	497	390	-0.006	0.135	0.135	0.966	96.646
$\mathbf{b}_{13}$	7.348	7.348	461	524	-0.004	0.131	0.131	0.968	1.058
$\mathbf{b}_{14}$	8.409	8.409	573	765	-0.001	0.115	0.115	0.958	0.898
$\mathbf{b}_{15}$	11.200	11.200	962	1418	0.000	0.089	0.089	0.950	0.666
$\mathbf{b}_{16}$	13.919	13.919	1371	2234	0.002	0.074	0.074	0.955	0.543
$\mathbf{b}_{17}$	15.793	15.793	1640	2883	0.003	0.068	0.069	0.951	0.495
$\mathbf{b}_{18}$	16.766	16.766	1748	3211	0.001	0.067	0.067	0.955	0.482
$\mathbf{b}_{19}$	17.739	17.739	1825	3488	-0.002	0.066	0.066	0.952	0.472
$\mathbf{b}_{20}$	18.816	18.816	1881	3747	-0.005	0.067	0.067	0.945	0.466
$\mathbf{b}_{21}$	20.034	20.034	1917	3997	-0.008	0.066	0.067	0.950	0.460
`Uniform`								0.974	8.886
`WBATE`					0.001	0.034	0.034	0.964	5.288
`LBATE`					0.155	0.089	0.179	0.996	83.651

Table SA-110: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, Kink-Adaptive Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	7.749	7.749	513	675	-0.002	0.151	0.151	0.955	0.587
$\mathbf{b}_2$	7.482	7.482	490	638	0.001	0.151	0.151	0.954	0.587
$\mathbf{b}_3$	7.315	7.315	474	614	0.002	0.147	0.147	0.957	0.581
$\mathbf{b}_4$	7.264	7.264	467	604	-0.000	0.148	0.148	0.954	0.568
$\mathbf{b}_5$	7.290	7.290	465	602	-0.000	0.144	0.144	0.955	0.550
$\mathbf{b}_6$	7.350	7.350	463	600	0.001	0.138	0.138	0.957	0.534
$\mathbf{b}_7$	7.387	7.387	455	589	-0.000	0.138	0.138	0.953	0.525
$\mathbf{b}_8$	7.287	7.287	449	530	0.002	0.137	0.137	0.959	0.523
$\mathbf{b}_9$	7.245	7.245	476	447	0.010	0.136	0.137	0.955	0.519
$\mathbf{b}_{10}$	7.128	7.128	500	343	0.010	0.133	0.134	0.958	0.513
$\mathbf{b}_{11}$	7.317	7.317	563	261	0.002	0.124	0.124	0.961	0.473
$\mathbf{b}_{12}$	7.227	7.227	497	390	-0.006	0.135	0.135	0.953	0.522
$\mathbf{b}_{13}$	7.347	7.347	461	523	-0.004	0.131	0.131	0.959	0.512
$\mathbf{b}_{14}$	7.501	7.501	469	610	-0.000	0.130	0.130	0.956	0.505
$\mathbf{b}_{15}$	7.420	7.420	477	619	-0.001	0.136	0.136	0.950	0.513
$\mathbf{b}_{16}$	7.278	7.278	468	606	0.005	0.136	0.136	0.955	0.527
$\mathbf{b}_{17}$	7.253	7.253	465	601	0.007	0.142	0.142	0.947	0.542
$\mathbf{b}_{18}$	7.408	7.408	475	617	0.001	0.139	0.139	0.949	0.542
$\mathbf{b}_{19}$	7.779	7.779	502	661	-0.003	0.135	0.135	0.954	0.531
$\mathbf{b}_{20}$	8.244	8.244	530	710	-0.000	0.137	0.137	0.948	0.524
$\mathbf{b}_{21}$	8.777	8.777	556	759	-0.000	0.133	0.133	0.952	0.515
`Uniform`								0.964	0.817
`WBATE`					0.001	0.041	0.042	0.954	0.162
`LBATE`					0.211	0.091	0.230	0.995	0.895

Table SA-111: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, Unknown-Kink Bandwidth, Fuzzy Bandwidth Selector).

	h_0	h_1	N_{Co}	N_{Tr}	Bias	SD	RMSE	EC	IL
$\mathbf{b}_1$	16.600	16.600	1719	3154	-0.004	0.080	0.080	0.960	0.547
$\mathbf{b}_2$	16.166	16.166	1690	3035	-0.004	0.079	0.079	0.959	0.543
$\mathbf{b}_3$	15.882	15.882	1686	2931	-0.004	0.079	0.079	0.963	0.536
$\mathbf{b}_4$	15.852	15.852	1738	2864	-0.001	0.080	0.080	0.964	0.526
$\mathbf{b}_5$	16.025	16.025	1840	2807	0.005	0.081	0.081	0.969	0.510
$\mathbf{b}_6$	16.350	16.350	1980	2738	0.013	0.079	0.080	0.963	0.488
$\mathbf{b}_7$	16.274	16.274	2037	2515	0.019	0.079	0.081	0.956	0.480
$\mathbf{b}_8$	16.244	16.244	2091	2279	0.021	0.076	0.079	0.958	0.472
$\mathbf{b}_9$	16.123	16.123	2107	2009	0.018	0.073	0.075	0.963	0.470
$\mathbf{b}_{10}$	15.915	15.915	2087	1714	0.010	0.069	0.070	0.968	0.462
$\mathbf{b}_{11}$	16.264	16.264	2148	1536	0.001	0.065	0.065	0.969	0.458
$\mathbf{b}_{12}$	16.237	16.237	2145	1881	-0.009	0.070	0.071	0.969	0.467
$\mathbf{b}_{13}$	16.776	16.776	2209	2356	-0.016	0.072	0.073	0.963	0.452
$\mathbf{b}_{14}$	16.977	16.977	2166	2747	-0.014	0.074	0.076	0.964	0.452
$\mathbf{b}_{15}$	16.647	16.647	1996	2940	-0.004	0.079	0.079	0.964	0.471
$\mathbf{b}_{16}$	15.901	15.901	1757	2891	0.007	0.078	0.078	0.967	0.497
$\mathbf{b}_{17}$	15.588	15.588	1630	2851	0.010	0.077	0.077	0.953	0.516
$\mathbf{b}_{18}$	15.786	15.786	1612	2884	0.007	0.075	0.075	0.954	0.516
$\mathbf{b}_{19}$	16.349	16.349	1641	2994	0.003	0.076	0.076	0.955	0.512
$\mathbf{b}_{20}$	17.026	17.026	1658	3109	0.002	0.078	0.078	0.949	0.515
$\mathbf{b}_{21}$	17.757	17.757	1651	3197	0.001	0.079	0.079	0.956	0.520
`Uniform`								0.976	0.752
`WBATE`					0.003	0.030	0.030	0.948	0.164
`LBATE`					0.087	0.060	0.106	0.996	0.879

Table SA-112: Fuzzy Effects (Simulations, Quadratic Heteroskedastic, `rdrobust` Bandwidth, Fuzzy Bandwidth Selector).

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